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THE USE OF MULTILINEAR REGRESSION MODELS IN PATTERNED WATERFLOODS: PHYSICAL MEANING OF THE REGRESSION COEFFICIENTS

by

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Dedication

To:
Daniela
Agustín
Joaquín
Paula
Hugo
Ale
Ceci
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The University of Texas at Austin, 2005

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One of the reservoirs engineer’s mission is to predict the behavior of hydrocarbon producing assets. Once this ability is developed he/she will try to manage the “today” to maximize the future economic return of the asset. However, the techniques to predict future performance vary from an educated guess of an appropriate analogy to very complex numerical approximations. But what they all have in common is that they are analyzing performance in the past to say something about the future. Hence, most models rely on fitting or matching to historic data.

Albertoni (2002) proposed yet another approach using well rate fluctuations in waterfloods to predict interwell connectivity. He expressed the total fluid production
at a producer as a weighted linear combination of the injection rates at different injectors located in the same reservoir. The relationship between the weights and the formation geological characteristics were not clear. For example, injection wells with no hydraulic connection to a producer may still exhibit a significant or even a negative weight.

This work explores the physical meaning of the weights and proposes a new way to interpret them. In addition, the original model has been expanded and is now able to incorporate flowing bottomhole pressure fluctuations. These insights are used to better understand the underlying assumptions of the model used by Albertoni (2002) and to construct a procedure for incorporating production data into geostatistical permeability distribution models.

The new interpretation of the weights arises as an analogy between constrained regression and parallel flow from each injector to all the producers. The procedure shows that the weights can be interpreted as the ratio of inverse distance weighted average permeabilities of well pairs associated with each injector (transmissibilities). They can also be interpreted as individual injector-producer water allocation factors that would result if there were no other injectors. This has been confirmed with flow simulation.

Finally, a new water allocation model is proposed that, in combination with a water-oil ratio power-law model, has been used to regress oil rates with encouraging results in synthetic and real datasets.
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CHAPTER 1: INTRODUCTION

One of the reservoirs engineer’s mission is to predict the future behavior of hydrocarbon producing assets. Once this ability is developed he/she will try to manage the “today” to maximize the future economic return of the asset. Techniques to predict future performance vary from an educated guess based on an appropriate analogy to very complex numerical approximations. But what they all have in common is that they are analyzing performance in the past to say something about the future. Hence, most prediction models rely on calibration or matching to historic data.

Albertoni (2002) proposed yet another approach using historic well rate fluctuations to predict reservoir production obtaining interwell connectivities in the process. He expressed the total fluid production at a producer as a weighted linear combination of the injection rates at different injectors located in the same reservoir. The key conclusions are that injector-producer weights are (a) independent of average well rates, (b) appear to reflect geologic features, (c) require some filtering in the presence of dissipation, and (d) can be calculated with several statistical procedures. The best thing about the method is that it can be used on virtually any injection process because injection-production rate data exists on all projects.

The present work extends Albertoni’s approach by focusing on the physical information content of the regression weights, addressing some limitations, and envisioning their use as one more gadget in the reservoir engineer’s toolbox.
In particular, in Chapter 2 the physical content of the regression weights is explored to reveal an underlying behavior that informs about the relative magnitudes of transmissibility-like character. Furthermore, for the ideal case of a linear reservoir, if the reservoir properties were known, the weights could readily be forward calculated and the regression weights could be considered reservoir characterization at the inter-well scale.

Chapter 3 shows that the incorporation of operating conditions into the system results in an ordinary cokriging-like system that averts the constant bottomhole pressure restriction, provides additional information, and results in more robust regression weights. In particular, the regression model is now able to deal with shut-in periods of the producers.

Chapter 4 presents several regression models that attempt, within limits, to generate coefficients that mimic injector-producer water allocation or tracer response.

From a regression tool point of view, Chapter 5, makes use of a water-oil ratio versus volume-injected power-law to provide oil phase fit. Information conveyed in the oil phase rates is now put in use.

These techniques can be used independently to provide a quick characterization or they could be integrated into a more comprehensive characterization workflow. Chapter 6 provides interpretation guidelines and a non-exhaustive list of applications with, stress on geostatistical and multilinear regression weight integration, as a mean of incorporating information from the dynamical behavior into geostatistical realizations.
1.1. REGRESSION MODEL REVIEW

1.1.1. Definitions

It is important to recall the definition of the following terms that are used later in this work.

In probability, the expectation of a random variable $X$ is defined as the first order moment of a probability density distribution $f(x)$.

$$E(X) = \int_{-\infty}^{+\infty} x f(x) \, dx$$  \hspace{1cm} (1.1)

To estimate the expected value of a random variable the arithmetic mean of the samples is computed.

$$E(X) = \frac{1}{M} \sum_{m=1}^{M} x^{(m)}$$  \hspace{1cm} (1.2)

The variance is the second-order centered moment of a probability distribution $f(x)$.

$$\text{Var}(X) = \int_{-\infty}^{+\infty} [x - E(X)]^2 f(x) \, dx$$  \hspace{1cm} (1.3)

Using the expectation operator definition the variance of $x$ can also be expressed as:

$$\text{Var}(X) = \sigma^2_X = E\left\{[x - E(X)]^2\right\}$$  \hspace{1cm} (1.4)

The variance is the expected value of the square of the deviation of $x$ from its own mean. Alternatively it can be expressed as:

$$\text{Var}(X) = E(X^2) - E^2(X)$$  \hspace{1cm} (1.5)
For discrete samples the estimator is:

\[
\text{Var}(X) = \sigma_X^2 = \sum_{m=1}^{M} \left[ x^{(m)} - E(X) \right]^2
\]

(1.6)

Similarly, the covariance between X and Y is defined as:

\[
\text{Cov}(X, Y) = \sigma_{XY}^2 = E \left[ x - E(X) \right] \left[ y - E(Y) \right]
\]

(1.7)

In our models, the superscript m is a time level variable and M is the total number of data points (time intervals).

1.1.2. Notation

In the context of this work, we refer to the total, three-phase production rate in reservoir volumes simply as the “production rate”. Likewise, injection rates are also considered to be in reservoir rates. The injection and production rates are considered to be positive.

The letter “P” is indicates a production rate. Similarly, the letter “I” indicates an injection rate. Estimated rates are distinguished from actual rates by a “hat” (^) symbol over the former. The arithmetic mean is denoted with the “\(\bar{\cdot}\)” symbol. Regression weights are represented with the Greek letter “\(\beta\)”. Producer wells are identified with the “j” index and injection wells are identified with “i” index. For example:

- \(P_j\) is the rate of producer j.
- \(\hat{P}_j\) is the estimated rate of producer j.
- \(\bar{P}_j\) is the time-average rate of producer j.
- \(I_i\) is the rate of injector i.
- \(\beta_{ji}\) is the weight that relates injector i to producer j.
1.1.3. Multi Linear Regression Model

In the multi-linear regression (MLR) model proposed by Albertoni (2002) the production rate is estimated as a linear combination of the injection rates plus an independent term to correct for bias.

\[ \hat{P}_j = \sum_i \beta_{ji} I_i + \beta_{j0} \]  

(1.8)

The system is solved by minimizing the variance of the prediction error:

\[ \text{Min} \left[ \text{Var}(\hat{P}_j - P_j) \right] = \text{Min} \left\{ E \left[ (\hat{P}_j - P_j)^2 \right] - E^2 \left[ (\hat{P}_j - P_j) \right] \right\} \]  

(1.9)

To minimize the variance, we set the derivative with respect to each of the variables \( \beta_{ij} \) equal to zero:

\[ \frac{\partial}{\partial \beta_{ij}} \left[ \text{Var}(\hat{P}_j - P_j) \right] = 0 \]  

(1.10)

The resulting system of linear equations in matrix form is:

\[
\begin{pmatrix}
\sigma_{I1}^2 & \sigma_{I2}^2 & \cdots & \sigma_{II}^2 \\
\sigma_{21}^2 & \sigma_{22}^2 & \cdots & \sigma_{2I}^2 \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{I1}^2 & \sigma_{I2}^2 & \cdots & \sigma_{II}^2
\end{pmatrix}
\begin{pmatrix}
\beta_{j1} \\
\beta_{j2} \\
\vdots \\
\beta_{jI}
\end{pmatrix}
= 
\begin{pmatrix}
\sigma_{j1}^2 \\
\sigma_{j2}^2 \\
\vdots \\
\sigma_{jI}^2
\end{pmatrix}
\]  

(1.11)

or in a more compact vector notation:

\[ C_{II} \beta = C_{PI} \]  

(1.12)

where \( C_{II} \) is the injector-injector covariance matrix
\[
C_\Pi = \begin{pmatrix}
\sigma_{11}^2 & \sigma_{12}^2 & \cdots & \sigma_{1I}^2 \\
\sigma_{21}^2 & \sigma_{22}^2 & \cdots & \sigma_{2I}^2 \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{I1}^2 & \sigma_{I2}^2 & \cdots & \sigma_{II}^2
\end{pmatrix}
\]

(1.13)

\(C_{PI}\) is the producer-injector covariance vector

\[
C_{PI} = \begin{pmatrix}
\sigma_{j1}^2 \\
\sigma_{j2}^2 \\
\vdots \\
\sigma_{jI}^2
\end{pmatrix}
\]

(1.14)

\(\beta\) is a vector containing the weights

\[
\beta = \begin{pmatrix}
\beta_{j1} \\
\beta_{j2} \\
\vdots \\
\beta_{jI}
\end{pmatrix}
\]

(1.15)

To obtain the weights, the injector-injector covariance matrix needs to be inverted.

\[
\beta = C_\Pi^{-1} C_{PI}
\]

(1.16)

The independent term \(\beta_{j0}\) is computed as a bias correction:

\[
E(P_j) = E(\hat{P}_j)
\]

(1.17)

\[
\beta_{j0} = E(P_j) - E\left(\sum_{i} \beta_{ji} I_i\right)
\]

(1.18)

### 1.1.4. Average Balanced Model (ABMLR)

For ABMLR, there is no bias term in the system (\(\beta_{j0}=0\)), bias being accounted for by a separate condition or equations. This model requires that the estimated production rates be, on average, balanced with the weighted injection rates.
\[
\begin{align*}
\hat{P}_j &= \sum_i \beta_{ji} I_i \\
\bar{P}_j &= \sum_i \beta_{ji} \bar{I}_i
\end{align*}
\] (1.19)

The balance constraint is achieved by introducing a Lagrange multiplier \(2\mu\) into the objective function to be minimized.

\[
\text{Min} \left[ \text{Var}(\hat{P}_j - P_j) + 2\mu \left( \sum_i \beta_{ji} \bar{I}_i - \bar{P}_j \right) \right]
\]

To minimize this function, we again set the derivative with respect to each of the variables \(\beta_{ij}\) and \(\mu\) equal to zero. This leads to the following system of equations one for each producer:

\[
\begin{align*}
\frac{\partial}{\partial \beta_{ij}} \left[ \text{Var}(\hat{P}_j - P_j) + 2\mu \left( \sum_i \beta_{ij} \bar{I}_i - \bar{P}_j \right) \right] &= 0 \\
\frac{\partial}{\partial \mu} \left[ \text{Var}(\hat{P}_j - P_j) + 2\mu \left( \sum_i \beta_{ij} \bar{I}_i - \bar{P}_j \right) \right] &= 0
\end{align*}
\]

which leads to the following system in matrix form:

\[
\begin{pmatrix}
\sigma_{i1}^2 & \sigma_{i2} & \cdots & \sigma_{iI}^2 & \bar{I}_1 \\
\sigma_{21}^2 & \sigma_{22} & \cdots & \sigma_{2I}^2 & \bar{I}_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\sigma_{I1}^2 & \sigma_{I2}^2 & \cdots & \sigma_{II}^2 & \bar{I}_I
\end{pmatrix}
\begin{pmatrix}
\beta_{j1} \\
\beta_{j2} \\
\vdots \\
\beta_{jI} \\
\mu
\end{pmatrix} = \begin{pmatrix}
\sigma_{j1}^2 \\
\sigma_{j2}^2 \\
\vdots \\
\sigma_{jI}^2 \\
\bar{P}_j
\end{pmatrix}
\]

Equation (1.21) can be expressed in a more compact notation using the definitions below.

\[
\begin{pmatrix}
C_{II} & \bar{I} \\
-I^T & 0
\end{pmatrix}
\begin{pmatrix}
\beta \\
\mu
\end{pmatrix} = \begin{pmatrix}
C_{PI} \\
\bar{P}_j
\end{pmatrix}
\]

(1.22)

where,
$C_{II}$ is the injector-injector covariance matrix

$C_{PI}$ is the producer-injector covariance vector for producer $j$

$\beta$ is a vector containing the weights

$\mu$ is the Lagrange multiplier

$\bar{I}$ is a vector containing the time average injection rates

$\bar{P}_j$ is the time-averaged rate of the producer being estimated.

### 1.1.5. Instantaneously Balanced Model (IBMLR)

The instantaneously balanced model (IBMLR) proposed by Albertoni is similar to the MLR model. The production rate is described as a fractional contribution from each of the injectors with no bias term and a separate balance condition. The balance condition in this case is stronger than for ABMLR requiring the waterflood to be in balance at every sampled time. IBMLR is, thus, best used when the waterflood is in balance (the field-wide injection rate is approximately equal to the total production rate) at every time:

$$\hat{P}_j = \sum_i \beta_{ji} I_i$$  \hspace{1cm} (1.23)

The instantaneous balance condition is:

$$\sum_{j=1}^{N} \hat{P}_j = \sum_{i=1}^{I} I_i$$  \hspace{1cm} (1.24)

$$\sum_{j=1}^{N} \sum_{i=1}^{I} \beta_{ij} I_i = \sum_{i=1}^{I} I_i$$  \hspace{1cm} (1.25)

which can be written as:
\[ \sum_{i=1}^{I} I_i \left( \sum_{j=1}^{N} \beta_{ij} \right) = \sum_{i=1}^{I} I_i \]  
(1.26)

Hence, the balance condition for each injector is given by:
\[ \sum_{j=1}^{N} \beta_{ij} = 1 \]  
(1.27)

IBMLR is the MLR system with an added set of constraints:
\[ \begin{cases} 
\hat{P}_j = \sum_{i} \beta_{ji} I_i & j = 1,...,N \\
\sum_{j=1}^{N} \beta_{ij} = 1 & i = 1,...,I 
\end{cases} \]  
(1.28)

This system resembles an ordinary Kriging (OK) system of equations. However, while in the OK system the weights that are constrained to sum to one are those corresponding to the estimation location (producer); here we are specifying that all weights associated to a given data location (injector) sum to one.

Note that while the MLR system can be solved for one producer at a time, the IBMLR system couples all injector and producers through the balance constraint; thus the system must be solved simultaneously for all producers. The constraints (one for each injector) are introduced in the system of equations again by means of Lagrange multipliers.

The system is solved by minimizing the following objective function.
\[ \text{Min} \left[ \sum_{j} \left( \text{Var}(\hat{P}_j - P_j) \right) + \sum_{i} 2\mu_i \left( \sum_{j=1}^{N} \beta_{ij} - 1 \right) \right] \]  
(1.29)

Recall that
\[ \text{Var}(\hat{P}_j - P_j) = E \left[ (\hat{P}_j - P_j)^2 \right] - E^2 \left[ (\hat{P}_j - P_j) \right] \]  
(1.30)
To minimize the variance, we again set the derivative with respect to each of the variables $\beta_{ij}$ and $\mu_i$ equal to zero:

$$\frac{\partial}{\partial \beta_{ij}} \left[ \sum_j \left( \text{Var}(\hat{P}_j - P_j) \right) + \sum_i 2\mu_i \left( \sum_{j=1}^{N} \beta_{ij} - 1 \right) \right] = 0$$

(1.31)

$$\frac{\partial}{\partial \mu_i} \left[ \sum_j \left( \text{Var}(\hat{P}_j - P_j) \right) + \sum_i 2\mu_i \left( \sum_{j=1}^{N} \beta_{ij} - 1 \right) \right] = 0$$

(1.32)

These results in a $(N \times I)+I$ by $(N \times I)+I$ system (recall that the MLR system is $I$ by $I$):

$$\begin{pmatrix} C_{\text{II}} & 0 & \ldots & \bar{I} \\ \bar{0} & C_{\text{II}} & 0 & \bar{I} \\ \vdots & \bar{0} & \ddots & \vdots \\ \bar{I} & \bar{I} & \ldots & 0 \end{pmatrix} \begin{pmatrix} \beta \\ \bar{\mu} \end{pmatrix} = \begin{pmatrix} C_{\text{IP}_1} \\ \vdots \\ C_{\text{IP}_j} \end{pmatrix}$$

(1.33)

where:

$C_{\text{II}}$ is the injector-injector covariance matrix

$C_{\text{IP}_j}$ is the producer $j$-injector covariance vector

$\bar{0}$ is an $I \times I$ matrix containing zeroes

$\bar{I}$ is an $I \times I$ identity matrix (not to be confused with average injection rates)

$\bar{\beta}$ is the vector containing the $N \times I$ weights

$\bar{\mu}$ is the vector containing $I$ Lagrange multipliers

$\bar{I}$ is the vector containing $I$ ones.

1.2. MODEL DISCUSSION REVISITED

The purpose of this section is to provide a better understanding of the behavior of the weights $\beta_{ji}$ with respect to the statistical characteristics of the input.
data and the relationship between the three systems (MLR, ABMLR, IBMLR) making use of analytical statistics.

We start by showing how IBMLR ensures that the input injection variance is transferred to the output production rates.

Making use of Eq. (1.23), the variance of the field-wide production rate can be expressed as:

\[
\text{Var} \left( \sum_j \hat{P}_j \right) = \text{Var} \left( \sum_j \sum_i \beta_{ji} I_i \right)
\]  
(1.34)

Summation indexes can be swapped and the terms rearranged this gives:

\[
\text{Var} \left( \sum_i \sum_j \beta_{ij} I_i \right) = \text{Var} \left( \sum_i I_i \sum_j \beta_{ji} \right)
\]

Making use of the IBMLR balance condition Eq. (1.27) results in:

\[
\text{Var} \left( \sum_j \hat{P}_j \right) = \text{Var} \left( \sum_i I_i \right)
\]  
(1.35)

The condition that the sum of the injector’s weights sum to one in the IBMLR model implies that the variance of the system is preserved. In other words, the variance from the injectors is transferred to the producers. Signals are neither amplified nor reduced.

This result suggests that the IBMLR weights do not exhibit a behavior observed in the MLR weights whereby injectors with large variance consistently obtain lower weights than those with small variance. In the latter case, their overall variance is reduced.
1.2.1. Lagrange Multipliers

Lagrange multipliers represent the price to pay\(^1\) for a unit deviation of the constraint in terms of the objective function.

For the IBMLR objective function Eq. (1.29) is the sum of the variances of the producers. The Lagrange multiplier represents the increase or decrease in total variance caused by an increase of one unit in constraint Eq. (1.27). Such a change in the constraint, associated to a particular injector, would be equivalent to duplicating the injector since the sum of the weights for the injector would now equal two. In other words, the effect of doubling an injection rate is assessed.

We would expect injectors with larger variance to have more impact on the sum of the variance of the producers than those with small variance for a given deviation of the constraint. Thus, we would expect larger Lagrange multiplier for those injectors with larger variance. This behavior is shown in Figure 1-1 for synthetic cases 15 and 250 that are to be detailed in the next chapter.

\[\text{Figure 1-1 Lagrange multiplier vs. variance. Left: case 250 consisting of 25 injectors. Right: case 15 consisting of 5 injectors.}\]

\(^1\) Actually, half the price to pay since our Lagrange multiplier is 2\(\mu\) instead of \(\mu\).
1.2.2. Variance of the MLR weights

The variance of the weights is a measure of the reliability of the estimated weights. In the case of the simpler unconstrained MLR system we can analyze the variance of the weights. Matrix notation will be used for simplification.

Consider the matrix $\mathbf{I}$ consisting of vector columns (one for each injector) containing the injection rate data and the vector $\mathbf{P}_j$ containing a column of production rate data for producer $j$.

$$
\begin{align*}
\mathbf{I} &= \begin{pmatrix}
\mathbf{i}_{1}^{t=1} & \cdots & \mathbf{i}_{I}^{t=1} \\
\vdots & \ddots & \vdots \\
\mathbf{i}_{1}^{t=M} & \cdots & \mathbf{i}_{I}^{t=M}
\end{pmatrix} \\
\mathbf{P}_j &= \begin{pmatrix}
\mathbf{p}_{j}^{t=1} \\
\vdots \\
\mathbf{p}_{j}^{t=M}
\end{pmatrix}
\end{align*}
$$

(1.36) (1.37)

An estimate of the injector-injector covariance matrix (1.13) can be expressed as:

$$
\mathbf{C}_{II} = \frac{1}{M} (\mathbf{I}^T \mathbf{I})
$$

(1.38)

and the producer–injector covariance matrix (1.14) as:

$$
\mathbf{C}_{PI} = \frac{1}{M} \mathbf{I}^T \mathbf{P}_j
$$

(1.39)

Thus the weights for producer $j$ can be estimated as:

$$
\hat{\beta} = (\mathbf{I}^T \mathbf{I})^{-1} \mathbf{I}^T \mathbf{P}_j
$$

(1.40)

The production rate can be expressed as a function of the error free weights $\beta_o$ plus a zero mean (probably measurement) error term ($\varepsilon_o$):

---

2 For centered moments the mean is subtracted from the vector of rates in $\mathbf{I}$ and $\mathbf{P}_j$. 
\[ P_j = I \beta_o + \epsilon_o \]  

(1.41)

By replacing Eq. (1.41) into Eq. (1.40) the weight error as a function of \( \epsilon_o \) can be calculated:

\[
\hat{\beta} = \left( I^T I \right)^{-1} I^T (I \beta_o + \epsilon_o) \\
\hat{\beta} = \left( I^T I \right)^{-1} I^T I \beta_o + \left( I^T I \right)^{-1} I^T \epsilon_o \\
\hat{\beta} = \beta_o + \left( I^T I \right)^{-1} I^T \epsilon_o \\
\hat{\beta} - \beta_o = \left( I^T I \right)^{-1} I^T \epsilon_o 
\]

(1.42)

Introducing the definition of the variance of the weights:

\[
\text{Var}(\hat{\beta}) = E \left[ \left( \hat{\beta} - \beta_o \right) \left( \hat{\beta} - \beta_o \right)^T \right] 
\]

(1.43)

Substituting the expression for error of the weights Eq. (1.42) into Eq. (1.43):

\[
\text{Var}(\hat{\beta}) = E \left[ \left( I^T I \right)^{-1} I^T \epsilon_o \left( \left( I^T I \right)^{-1} I^T \epsilon_o \right)^T \right] \\
\text{Var}(\hat{\beta}) = E \left[ \left( I^T I \right)^{-1} I^T \epsilon_o \epsilon_o^T I \left( I^T I \right)^{-1} \right] \\
\text{Var}(\hat{\beta}) = C_{II}^{-1} E \left[ \epsilon_o \epsilon_o^T \right] C_{II}^{-1} \\
\]

The expression \( E \left[ \epsilon_o \epsilon_o^T \right] \) represents the variance of the (measurement) error in the producer’s rate estimation \( \sigma_\epsilon^2 \):

\[
\text{Var}(\hat{\beta}) = C_{II}^{-1} \sigma_\epsilon^2 C_{II}^{-1} = \sigma_\epsilon^2 C_{II}^{-1} C_{II} C_{II}^{-1} \\
\]

Finally:

\[
\text{Var}(\hat{\beta}) = \sigma_\epsilon^2 C_{II}^{-1} 
\]

(1.44)

Hence the covariance of the weights is proportional to the measurement noise and to the inverse of the injector-injector covariance matrix. The latter is large in
particular when co-linearity exists between the injectors (Yousef, in progress) or when one or more of the injectors present very small fluctuations. Note that expression Eq. (1.44) includes covariances terms between weights that provide information about the interrelation of the weights.

1.2.3. Minimizing Measurement Error Impact

We have just shown how measurement error affects the accuracy of the regression weights in expression Eq. (1.44). Albertoni also studied the impact of measurement errors by performing a sensitivity study on the coefficient of determination $R^2$ for the MLR and his asymmetry coefficient $A$. Albertoni’s asymmetry coefficient $A$ is the square root of the sum of the variances of the weights of the symmetric group, weighted by the number of well pairs in each group.

$$A = \sqrt{\frac{8\text{Var}_a(\beta) + 8\text{Var}_b(\beta) + 4\text{Var}_c(\beta)}{20}}$$

(1.45)

where $a$, $b$, $c$ are symmetry groups.

In section 2.3.3.e we discuss why we expect injector-producer pairs within symmetry groups to have similar weights in a homogeneous field. Hence, if all pairs within a group have similar weights, we expect the asymmetric coefficient $A$ to approximately zero.

Figure 1-2 shows the result of the sensitivity of $R^2$ and $A$ to the standard deviation of a zero mean error of the historic rates dataset. According to the figure, in a homogeneous low compressibility reservoir we expect an error-free dataset to have a large $R^2$ (possibly equal to one) and a small (possibly zero) asymmetry coefficient $A$. As the error increases $R^2$ decreases and $A$ increases.
To minimize the detrimental effects of (measurement) errors in the dataset we propose to process the original dataset in such a fashion that the resulting dataset is error free. We are assuming that we are dealing with a zero mean type error.

General procedure:

- Shuffle the time level of the original dataset to obtain a number of datasets where the time ordering has been randomized.
Stack or average across (the time levels of) the randomized datasets. Since we are assuming zero mean error, the resulting dataset should have a higher signal to noise ratio.

Perform the regression on the resulting dataset.

The consistency of the resulting weights can be checked by repeating the procedure a several times and calculating the variance of the weights. A large variance might be an indication that either the error does not have mean zero or that the compressibility of the fluid is relatively large and thus the time independence assumed by MLR, ABMR and IBMLR is being violated. In the latter case, filtered models might be necessary (Albertoni 2002, Yousef, in progress)

### 1.2.4. Stability of the MLR System of Equations

It is useful to have an indicator of how amenable a particular dataset is to MLR analysis. We resort to the “Condition Number” from matrix analysis theory (El Ghaoui, 2002), which states that:

For a linear system in matrix form:

\[ Ax = b \]  \hspace{1cm} (1.46)

The relative error for a computed solution of the vector of weights \( \mathbf{x} \) is bounded according to:

\[ \frac{\| \Delta \mathbf{x} \|}{\| \mathbf{x} \|} \leq \epsilon \kappa(\mathbf{A}) \]  \hspace{1cm} (1.47)

where:

\[ \epsilon \ll 1 \]

\( \| \cdot \| \) is the norm operator
and
\[ \kappa(A) = \|A\| \|A^{-1}\| \] (1.48)
is the condition number.
\[ \kappa(A) \gg 1 \] indicates that the system of equations (1.46) is ill conditioned; small changes in input data produce large changes in output \( \mathbf{x} \). In other words, large departures from the true solution \( \mathbf{x}_0 \) satisfy Eq. (1.46) within tolerance \( \varepsilon \).

If \( A \) is symmetric like in the case of the covariance matrix \( C_{\mathbf{H}} \), the conditioning number can be expressed:
\[ \kappa(A) = \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}} \] (1.49)
where \( \lambda_{\text{max}} \) and \( \lambda_{\text{min}} \) are the maximum and minimum eigenvalues of matrix \( A \) respectively.

Thus, the condition number \( \kappa \) Eq. (1.49) could be used as quality indicator of the dataset in terms of the accuracy that can be expected from the arising weights. Note that as \( \lambda_{\text{min}} \) tends to zero \( \kappa \) tends to infinity; the presence of one very small eigenvalue is an indication of a co-linear or near singular system and an immediate cause for concern.

1.2.5. Relation between MLR and ABMLR weights to IBMLR weights

Assuming a balanced system such as the synthetic fields (Synfields) used by Albertoni (2002) and throughout this work, if MLR is used as a regression model, it is likely that the sum of the weights for the injectors will not sum to one. This would create an artificial surplus or deficit of volume in the system (since it is balanced) that
is accounted for by the constant term $\beta_o$. In other words, a bias\(^3\) in the injector’s weight is transferred to the bias term of the MLR system. This effect has been particularly noticeable in data sets where the variances of each injector differ quite markedly, such as systems where injectors have frequent shut-in periods.

For the ABMLR system, the bias term is zero; hence, any imbalance, arising from the fact that the weights are not constrained to sum to one, has to be accounted by the injector weights themselves in such a fashion that the imbalances from each injector cancel out. This causes further departure from the unbiased estimates.

To analyze this we propose the following bias model where a bias associated to each injector is affecting the weights:

$$w_{ji} = \beta_{ji} + b_i$$  \hspace{1cm} (1.50)

where $\beta_{ji}$ is the bias-free or true weight and $b_i$ the constant weight bias for the injector.

The regression model is:

$$P_j = \sum_i w_{ji} I_i + \beta_{jo}$$  \hspace{1cm} (1.51)

where $\beta_{jo} \neq 0$ for MLR and $\beta_{jo} = 0$ for ABMLR and IBMLR.

Substituting Eq. (1.50) into Eq. (1.51):

$$P_j = \sum_i \left( \beta_{ji} + b_i \right) I_i + \beta_{jo} = \sum_i \beta_{ji} I_i + \left( \sum_i b_i I_i + \beta_{jo} \right)$$  \hspace{1cm} (1.52)

Note that the term $\sum_i b_i I_i$ behaves like an additional bias that is common to all producers.

---

\(^3\) In this section, “bias” of a weight is assumed to be with respect to the IBMLR weight.
Assuming we apply model Eq. (1.51) to a balanced reservoir and obtain a non-zero bias term, and since we expect the total bias (from Eq. (1.52)) to be zero for such a system, then:

\[ \beta_{jo} + \sum_i b_i I_i = 0 \]  \hspace{1cm} (1.53)

\[ \beta_{jo} = -\sum_i b_i I_i \]  \hspace{1cm} (1.54)

Thus, the bias term \( \beta_o \) in the MLR model, for a balanced system, introduces bias \((b_i)\) into the injector weights. Moreover, a common bias term for all producers on a balanced reservoir would be an indication of bias in the injection weights.

For ABMLR, the bias term \( \beta_o \) is zero by definition; hence we would expect the bias in the injector weights to have some positive and negative values such that they cancel:

\[ \sum_i b_i I_i = 0 \]  \hspace{1cm} (1.55)

The relationship between MLR and ABMLR with the IBMLR weights can be quantified in the following way:

For each injector the sum all it’s weights is a given number \( C_i \):

\[ \sum_j w_{ji} = C_i \]  \hspace{1cm} (1.56)

Substituting Eq.(1.50) into Eq.(1.56):

\[ \sum_j (\beta_{ji} + b_i) = C_i \]  \hspace{1cm} (1.57)

\[ \sum_j \beta_{ji} + \sum_j b_i = C_i \]  \hspace{1cm} (1.58)

Recall IBMLR constraint on the weights Eq.(1.27) and replace into Eq.(1.58):

\[ 1 + \sum_j b_i = C_i \]  \hspace{1cm} (1.59)
Assuming a common bias term for all producers, the summation simplifies into a multiplication:

\[ 1 + N \cdot b_i = C_i \]  
\[ b_i = \frac{C_i - 1}{N} \]

(1.60)  
(1.61)

where \( N \) is the number of producers associated to a given injector \( i \).

MLR and ABMLR weights (in this case generalized by \( w_{ji} \) Eq. (1.50)) can be approximated to the IBMLR weights by subtracting one from the sum of the weights \( (C_i) \) of a particular injector and dividing by the number of producers associated with that injector. A numerical verification is presented in section 2.5.2

1.2.6. Open Versus Closed Reservoirs Discussion

Throughout this work we will be applying these regression models on data sets generated by a numerical reservoir simulator. The extents of these numerical reservoirs are defined very clearly and undoubtedly classify as closed reservoirs.

However, it is tempting and sometimes otherwise impractical to apply a given regression model on subset of the field. This would cause the section of reservoir under analysis to have open boundaries. Although it might be argued that if the section is cut along a path connecting injectors these would act as barriers to flow (dividing waters), these injectors are not barriers for pressure.

More importantly, when a reservoir is to be divided in this manner, there are several ways to handle the rate of the injectors located on the boundaries. The usual practice of dividing the rates by two of those injectors located on lateral boundaries and dividing by four the rate of those located on a corner is arbitrary. On the other
hand, using the injection rates without allocation would make the system to become overbalanced, even if it was balanced on a pattern-by-pattern or field-wide basis. From a regression point of view we are injecting more variance than is presumably, necessary. We have seen, in the previous section, that imbalances either in terms of rates or in terms of variances are manifested by biases in the regression weights thus rendering their physical information useless.

An educated choice would, thus, be made by the analyst between performing a full field analysis or partial field analysis, and regarding the allocation of water of boundary injectors.

1.2.7. Conclusions

The IBMLR model is more robust than the MLR model with respect to datasets that have dissimilar magnitudes of variance in the injection rates. Lagrange multipliers are a measure of how strongly a deviation from the constraints affects the total variance of the predictions.

The accuracy of the weights decreases proportionally to the measurement noise and to the inverse of the injector-injector covariance matrix.

The stability of the regression model can be determined by calculating the ratio of the maximum to minimum eigenvalues of the injector-injector covariance matrix.

MLR and ABMLR weights can be approximated to the IBMLR weights by subtracting one from the sum of the weights of a particular injector and dividing by the number of producers associated to the injector.
In particular, if a reservoir is balanced:

For the MLR model, the bias contribution from each injector \((b_i I_i)\) is such that they sum to the constant bias term \((\beta_{jo})\). Moreover, the bias term \(\beta_{jo}\) is expected to be the same for all producers.

For the ABMLR model, the bias contribution from each injector \((b_i I_i)\) is such that they sum to zero.
CHAPTER 2: PHYSICAL MEANING OF THE REGRESSION WEIGHTS

In this chapter a physical interpretation of the weights is achieved by analogy to a physical system that results in a similar system of equations as the IBMLR model. The arising interpretations are tested by performing sensitivity on a number of reservoir simulator models.

2.1. PARALLEL FLOW ANALOGY

Consider a single injector injecting a rate $I_i$ and $j=1,…N$ producers producing at rate $P_j$ connected by arbitrary paths as depicted in Figure 2-1 with injector contribution rates $q_{ji}$ being transported along each path.

![Figure 2-1 Schematic depicting injector $I_i$ feeding $q_{ji}$ rates into producers $P_1$ through $P_j$.](image-url)
The physical system is rate balanced. The total injection rate is the sum of the injector’s contributions:

\[ I_i = \sum_j q_{ij} \]  

(2.1)

and considering the contribution of other possibly present injectors:

\[ P_j = \sum_i q_{ij} \]  

(2.2)

By inspection of Eq. (1.23) and Eq. (2.2), the two systems would be the same if

\[ q_{ij} = \beta_{ij} I_i \]  

(2.3)

Combining Eq. (2.1) and Eq. (2.3) gives the following condition for which further results are valid:

\[ I_i = \sum_j \beta_{ij} I_i \]

Rearranging

\[ \sum_j \beta_{ij} = 1 \]  

(2.4)

which is equivalent to Albertoni’s instantaneously balanced multivariate linear regression (IBMLR) balance condition.

Physical interpretations are now straightforwardly obtained.

From Eq. (2.3) we obtain:

\[ \beta_{ij} = \frac{q_{ij}}{I_i} \]  

(2.5)

Equation (2.5) states that the \( \beta_{ij} \) weight represents the ratio of the contribution rate established between an injector (\( I_i \)) – producer (\( P_j \)) pair to the total rate injected by said injector.
Likewise if transients are shorter than the sampling interval (typically one month) steady-state flow equations could be applied. For parallel flow in a linear system in which Darcy’s law applies:

\[ q_{ij} = \frac{k_{ij} A_{ij} (P_{wf}^i - P_{wf}^j)}{\mu L_{ij}} = T_{ij} \left( P_{wf}^i - P_{wf}^j \right) \]  

(2.6)

where

- \( k_{ij} \) is the effective average permeability for the flow path.
- \( A_{ij} \) is the area open to flow. Presumably flow geometry dependent.
- \( L_{ij} \) is the length of the path.
- \( \mu \) is the viscosity of the fluid (assumed constant).
- \( P_{wf}^i \) is the injection pressure (bottomhole).
- \( P_{wf}^j \) is producer flowing bottomhole pressure at the same datum as \( P_i \).
- \( T_{ij} \) is the transmissibility between injector \( i \) and producer \( j \).

Equation (2.6) can be substituted into equations (2.5), and (2.1) to obtain an expression of the IBMLR weights in terms of the Darcy equation variables.

\[ \beta_{ji} = \frac{A_{ji} k_{ji} (P_{wf}^i - P_{wf}^j)}{\mu L_{ji}} = \frac{T_{ji} \left( P_{wf}^i - P_{wf}^j \right)}{\sum_j T_{ji} \left( P_{wf}^i - P_{wf}^j \right)} \]  

(2.7)

We will be referring to Eq. (2.7) and it’s simplification or modification as the *physical estimates* of the weights.
As expected, this result suggests that the weights contain coupled information about rock properties (k), flow geometry (A, L), fluid properties (µ) and operating conditions of the wells (P_{i,wf}, P_{j,wf}). This analogy will help assimilate the resulting regression coefficient in physical terms.

2.1.1. Physical Interpretations

Equation (2.5) suggests that the weights can be thought as the ratio of the flow rate between an injector and any one producer to the total injector’s rate. The physical framework proposed that all contributions act independently; hence the weight would represent the amount of water allocated from a particular injector to a particular producer in the absence of other injectors. This is similar to the superposition concept commonly used in well testing models.

By assuming a transport model for these rates, we obtained Eq. (2.7). In this case we chose a linear model acknowledging that this might be an approximation to a nonlinear problem. Note that according to, Eqs. (2.5) or (2.7), the weights should lie between zero and one.

Thus, the IBMLR coefficients are analogous to the following physical concepts:

–Weighted transmissibility ratios as suggested by equation (2.7)
–Independent injection allocation as suggested by equation (2.5)

In addition, interesting relationships, as shown in the following sections, arise for particular cases.
a. Inverse Distance Weighting

In a homogeneous isotropic and balanced reservoir, the $\beta$ coefficients should only reflect geometry. If

\[ k_{ij}=k, \quad A_{ji}=A, \quad p_{wf}^{ji}=\text{BHP} \quad \text{(constant for all producers)} \]

then for such a system we should expect the $\beta$s to be:

\[ \beta_{ij} = \frac{1}{\sum_j \frac{1}{L_{ij}}} \]

(2.8)

Note that the sum is over the producers. The distances are from one particular injector to all producers. This has the form of a popular mapping interpolation technique known as inverse distance weighting (IDW).

More generally for a non-homogeneous reservoir:

\[ \beta_{ji} = \frac{A_{ji} \cdot k_{ji}}{\sum_j \frac{A_{ji} \cdot k_{ji}}{L_{ji}}} \]

(2.9)

b. Weight Ratios

For the case of a symmetrical pattern, such as a 5-spot, where flow geometries are expected to be similar (Figure 2-2), the ratio of weights for a given injector and any two producers within the same pattern can be estimated by:
Therefore for a given injector within a symmetric pattern, the ratio of weights between any two neighboring producers gives information about the ratio of average permeabilities.

\[
\beta_{ji} = \frac{A_{ji} k_{li}}{L_{ji}} = \frac{k_{li}}{k_{2i}}
\]

(2.10)

Furthermore, the relative weight of any injector-producer pair with respect to the weights of the entire pattern can be approximated by the ratio:

\[
R_{\beta_{ji}} = \frac{\beta_{ji}}{\sum_{j \text{Pattern}} \beta_{ji}}
\]

(2.11)

where, \(j_{\text{Pattern}}\) is a list of those producers that belong to injector i’s pattern.

Using Eq. (2.9), this ratio can be physically approximated by:
which for similar flow geometries further simplifies to:

\[
R_{k_{ji}} = \frac{A_{ji} \cdot k_{ji}}{\mu L_{ji}} \sum_{j_{Pattern}} \frac{1}{\mu L_{ji}}
\]

(2.12)

This ratio contains significant physical information. For an isolated five spot pattern, this ratio represents the fraction of water the injector allocates into each producer. Furthermore, for an isolated and homogeneous five spot, we would expect this value to be equal to 0.25.

Equation (2.13) has potential use in permeability field inversion problems. It will be extensively used in chapter 6.

2.2. FROM PRODUCER RATES TO INJECTOR RATES

The cause and effect could be reversed and an injection rate regression from production rate data can be proposed with a linear relationship analog to equation (1.23):
The injection rate is now a linear combination of the production rates. The coefficients \( \rho_{ij} \) scale the production rates to minimize the variance of the error of the estimate.

By using, once more, the steady state parallel flow analogy:
\[
\hat{I}_i = \sum_j q_{ij} \quad (2.15)
\]
\[
q_{ij} = \rho_{ij} P_j \quad (2.16)
\]
thus,
\[
\rho_{ij} = \frac{q_{ij}}{P_j} \quad (2.17)
\]

By equating Eq. (2.16) and Eq. (2.3), we obtain the relationship between the regressed production coefficients and the regressed injection coefficients:
\[
\rho_{ij} = \beta_{ji} \frac{I_i}{P_j} \quad (2.18)
\]

These injection from production (IFP) coefficients result in the same IBMLR weights scaled by the ratio of injection to production rate of the corresponding pair of wells. Since the ratio of injection to production is likely to vary over time for any particular pair of wells, so would the IFP weights and hence multi-linear regression of the IFP model (Eq. (2.14)) to obtain constant coefficients would seem inappropriate.

To gain more insight into this issue, we will once more make use of the linear Darcy equation to model \( q_{ij} \) and of a balance condition:
Recall Eq. (2.6):
\[
q_{ij} = \frac{A_{ij} k_{ij} (P_{wf}^i - P_{wf}^j)}{\mu L_{ij}}
\]
The balance condition for the producer is:
\[ \sum_{i} q_{ij} = p_{j} \]  \hspace{1cm} (2.19)

To obtain the IFP coefficients, Eq. (2.6) is replaced into equations (2.19) and (2.17)

\[ \rho_{ij} = \frac{A_{ij} k_{ij} \left( p_{wf}^{i} - p_{wf}^{j} \right)}{\mu L_{ij}} \]

\[ \sum_{i} \frac{A_{ij} k_{ij} \left( p_{wf}^{i} - p_{wf}^{j} \right)}{\mu L_{ij}} \]  \hspace{1cm} (2.20)

The main difference in this expression with respect to equation (2.7) is that the normalizing sum of rates in the denominator is over the injectors. Each term of the summation will have a different \( p_i \) and thus the pressure difference term will not cancel out with the corresponding term in the denominator. This confirms that the \( \rho_{ij} \) coefficients would fluctuate as indicated by equation (2.18).

However this behavior true if the injection pressures fluctuate and production bottomhole pressures are approximately constant. If the reverse were true for a particular field, that is, constant and approximately the same injections pressures and fluctuating BHP in the producers, then the pressure terms in Eq. (2.20) would cancel out rendering the coefficients constant and thus the IFP model should lend itself better for a regression analysis. In addition, expression Eq. (2.20) could be used to calculate their values given prior knowledge of the permeability field and well location layout.

Expression Eq. (2.18) allows us to transition from one model to another. Moreover, in expression Eq. (2.18), IBMLR weights \( \beta_{ij} \), injection rates \( I_i \), and production rates \( P_j \) are known which allows the calculation of the IFP coefficients \( \rho_{ij} \).
2.3. NUMERICAL EXPERIMENTS I: TESTING THE ESTIMATES OF THE WEIGHTS

In this section numerical experiments are performed to test the validity of the previously derived relationships between the regression weights and physical magnitudes.

It is natural to start the tests in the same context in which they were derived, that is to say, linear one-dimensional models.

2.3.1. One Dimensional Experiments

Schlumberger’s Eclipse E100 black oil simulator was used to generate the synthetic datasets. The model consists of 31 cells in x direction, 1 cell in the y direction, and 5 cells in the vertical direction. Although strictly speaking the model is a two-dimensional cross-section, the reservoir properties are constant in the vertical direction and they will referred to as 1D models in this work. Moreover, flow in this direction is neglible. The end-point mobility ratio is close to unity. The porosity is assumed spatially constant and equal to 0.18.

<table>
<thead>
<tr>
<th>Table 2-1 Reservoir properties for cross-section cases.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>Number of cells</td>
</tr>
<tr>
<td>Cell Length [ft]</td>
</tr>
<tr>
<td>Total Length [ft]</td>
</tr>
<tr>
<td>Permeability [md]</td>
</tr>
</tbody>
</table>

33
Procedure:

1. Construct simulation
   a. Operating Conditions
      i. Injection rates are arbitrarily specified (Figure 2-3).

\[ \text{Figure 2-3} \quad \text{Injection rates used in one-dimensional numerical experiments.} \]

   ii. The producers are operated at a common and constant bottomhole pressure.

b. Fluid Properties

   Two-phase undersaturated oil and water properties have been summarized in Table 2-2. This small compressibility system is representative of mature waterfloods where most gas has come out of solution and has been displaced resulting in a low bubble point pressure oil.
Table 2-2 Water and oil properties

<table>
<thead>
<tr>
<th></th>
<th>Oil</th>
<th>Water</th>
</tr>
</thead>
<tbody>
<tr>
<td>Formation</td>
<td>1.07 RB/STB</td>
<td>1.01 RB/STB</td>
</tr>
<tr>
<td>Compressibility</td>
<td>5.00 mips</td>
<td>1.00 mips</td>
</tr>
<tr>
<td>Viscosity</td>
<td>2 cp</td>
<td>0.5 cp</td>
</tr>
</tbody>
</table>

c. Rock properties

For all cases, the relative permeabilities and capillary pressures shown in Figure 2-4 were used.

![Relative Permeability and Capillary Pressure Curves](image)

**Figure 2-4** Relative permeability and capillary pressure curves used throughout the simulations.

d. The well locations are specific for each case.

2. Collect production rates to build synthetic data set for regression.
3. Regress production rates to injection rates using IBMLR and obtain the regression weights $\beta_{ji}$.
4. Calculate physical estimate of weights according to equation (2.9).
5. Compare results.
a. *Case 1201: Homogeneous Reservoir, Equidistant Well Locations*

We start with an intuitive case. A producer is located at each end of the model and the single injector is located halfway between. Figure 2-5 shows the location of the wells within the model grid. Producers are identified by an upward pointing arrow and the letter “P” at the corresponding location. Likewise, a downward pointing arrow and the letter “I” represent the injector locations. Unless noted otherwise, producers and injectors are numbered from left to right. The symmetry of the problem suggests that the injected water rate should split evenly between both producers.

Since this is a homogeneous system, we have seen that physical estimate equation (2.7) simplifies to the inverse distance weighting (IDW) estimator Eq. (2.8):

$$
\beta_{ji} = \frac{1}{L_{ji}} \sum_j \frac{1}{L_{ji}}
$$

We apply this equation measuring the distance in terms of the number of cells:

$$
\beta_{11} = \frac{1}{(16-1)} + \frac{1}{15} = \frac{1}{15} + \frac{1}{15} = \frac{1}{2}
$$

$$
\beta_{21} = \frac{1}{(31-16)} + \frac{1}{15} + \frac{1}{15} = \frac{1}{15} + \frac{1}{15} = \frac{1}{2}
$$

*Figure 2-5* Case 1201: one injector and two equidistant producers. Well locations within the grid are represented by arrows. Producers and injectors are numbered from left to right.
Table 2-3 Case 1201. Regressed weights vs. inverse distance weighting estimator.

<table>
<thead>
<tr>
<th></th>
<th>IBMLR</th>
<th>IDW Estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{11}$</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>$\beta_{21}$</td>
<td>0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>

IDW values can be compared to the IBMLR regression results in Table 2-3. As expected from our insight into the behavior of the weights, both weights are equal. Moreover, since they should sum to one, they should be equal to one half.
b. Case 1200-1: Homogeneous Case, Asymmetric Well Locations

In this case the injector was moved off the center to test the IDW estimator in an asymmetric flow configuration.

![Diagram of Case 1201-1](image.png)

**Figure 2-6** Case 1201-1. Two producers and one injector in asymmetric configuration.

As in the previous case, the IDW estimator applies:

\[
\beta_{ji} = \frac{1}{L_{ji}} \sum_{j} \frac{1}{L_{ji}}
\]

\[
\beta_{11} = \frac{1}{9} + \frac{1}{21} = \frac{1}{9 + 21} = 0.70
\]

\[
\beta_{21} = \frac{1}{1} + \frac{1}{(31-10)} = \frac{1}{21} = 0.30
\]

**Table 2-4** Case 1201-1. Regressed weights vs. inverse distance weighting estimator.

<table>
<thead>
<tr>
<th></th>
<th>IBMLR</th>
<th>IDW Estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_{11} )</td>
<td>0.69</td>
<td>0.70</td>
</tr>
<tr>
<td>( \beta_{21} )</td>
<td>0.31</td>
<td>0.30</td>
</tr>
</tbody>
</table>

Intuitively we would expect the weight of the closest producer (located in cell one) to be greater. This is indeed the case. In addition, by using the IDW estimator we are able to predict successfully these values, as shown in Table 2-4, within 3.5% of the IBMLR weights.
c. Case 1201-2: Heterogeneous 1D numerical model

The next step is to add heterogeneity to the previous model to test the inverse distance weighted permeability (IDWK) estimator. The permeability between the injector and the far away producer was deliberately chosen to compensate for the greater distance (same k/L).

Figure 2-7 Case 1201-2. Two producers and one injector in asymmetric and heterogeneous configuration.

In this case equation (2.7) would simplify to

\[
\beta_{ji} = \frac{k_{ji}}{\sum_{j} k_{ji} L_{ji}}
\]

(2.21)

\[
\beta_{11} = \frac{1}{1 + \frac{2.3}{(31-10)}} = \frac{1}{9 + \frac{2.3}{21}} = 0.504 \approx 0.50
\]

\[
\beta_{21} = \frac{2.3}{(31-10)} = \frac{2.3}{21} = 0.496 \approx 0.50
\]

Table 2-5 Case 1201-1. Two producers and one injector in an asymmetric and heterogeneous configuration. Regressed weights vs. inverse distance weighting estimator.

<table>
<thead>
<tr>
<th></th>
<th>IBMLR</th>
<th>IDWK Estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\beta_{11})</td>
<td>0.50</td>
<td>0.50</td>
</tr>
<tr>
<td>(\beta_{21})</td>
<td>0.50</td>
<td>0.50</td>
</tr>
</tbody>
</table>
d. **Case 1202: Homogenous 1D numerical model – Two Injectors**

For the following two cases, the number of injectors is increased from one to two to the hypothesis that each injector contributes to the producer independently.

We start with a single producer located in cell one (Figure 2-8).

![Figure 2-8](image)

**Figure 2-8** Case 1202. Homogenous 1D numerical model. Two Injectors and one producer.

If the injectors were analyzed individually with respect to the single producer we would expect the weights to be equal to one. This is also arises directly from the constraint condition.

Indeed, the IDW ratio is trivial and the constraint on the IBMLR system forces the weights to be equal to one. Making use of the IDWK estimator Eq. (2.21):

\[
\beta_{ji} = \frac{k_{ji}}{L_{ji}} \quad \beta_{11} = \frac{1}{(16-1)} = 1 \quad \beta_{12} = \frac{1}{(31-16)} = 1
\]

**Table 2-6** Case1202. IBMLR vs. IDW estimator comparison.

<table>
<thead>
<tr>
<th></th>
<th>IBMLR</th>
<th>IDW Estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\beta_{11})</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>(\beta_{12})</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>
e. Case 1203-1: Homogenous 1D numerical model – Two producers and Two Injectors

A second producer is reintroduced and a similar analysis is performed.

Applying the IDW physical estimator Eq. (2.8) to the current configuration:

\[
\begin{align*}
\beta_{11} &= \frac{1}{\frac{1}{(9-1)} + \frac{1}{(31-9)}} = 0.73 \\
\beta_{21} &= \frac{1}{\frac{1}{(31-9)} + \frac{1}{(9-1)}} = 0.27 \\
\beta_{12} &= \frac{1}{\frac{1}{(16-1)} + \frac{1}{(31-16)}} = 0.50 \\
\beta_{22} &= \frac{1}{\frac{1}{(31-16)} + \frac{1}{(16-1)}} = 0.50
\end{align*}
\]

Table 2-7 Case1203-1. IBMLR vs. IDW estimator comparison.

<table>
<thead>
<tr>
<th></th>
<th>IBMLR</th>
<th>IDW Estimator</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{11}$</td>
<td>0.75</td>
<td>0.73</td>
</tr>
<tr>
<td>$\beta_{12}$</td>
<td>0.50</td>
<td>0.50</td>
</tr>
<tr>
<td>$\beta_{21}$</td>
<td>0.25</td>
<td>0.27</td>
</tr>
<tr>
<td>$\beta_{22}$</td>
<td>0.50</td>
<td>0.50</td>
</tr>
</tbody>
</table>

One more time, Table 2-7 confirms a good agreement between the actual regression weights and the physical IDW estimator.
f. **Case 1410: Branched reservoir: Heterogenous 1D numerical model – Four producers & One Injector:**

We introduce the linear five-spot model. Except for the I1 injector location, all flow is linear. Flow branches off from the injector location to the four producers. This case is used to simultaneously vary the permeability and length of each branch. The well locations, distances and permeabilities are shown on Figure 2-10. Permeabilities and the length of the branches are given as multiples of the values in branch I1-P1.

![Figure 2-10](image_url)

**Figure 2-10** Case 1410. Branched reservoir with one injector and four producers. Length and permeability values of the branches are given with respect to the I1-P1 branch. $l =$ length of I1-P1 branch; $k =$ permeability of I1-P1 branch.
Table 2-8 The relative values of permeability and length are used in the IDWK estimator.

<table>
<thead>
<tr>
<th></th>
<th>( I_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>( \beta_{11} = \frac{1}{1+\frac{6}{2}+\frac{1}{2}+1} = 0.181 )</td>
</tr>
<tr>
<td>P2</td>
<td>( \beta_{21} = \frac{6}{2}{\frac{1}{2}+\frac{1}{2}+1} = 0.545 )</td>
</tr>
<tr>
<td>P3</td>
<td>( \beta_{31} = \frac{1}{1+\frac{6}{2}+\frac{1}{2}+1} = 0.091 )</td>
</tr>
<tr>
<td>P4</td>
<td>( \beta_{41} = \frac{1}{1+\frac{6}{2}+\frac{1}{2}+1} = 0.181 )</td>
</tr>
</tbody>
</table>

Table 2-9 Case1410. IBMLR vs. IDWK estimator comparison.

<table>
<thead>
<tr>
<th></th>
<th>IBMLR</th>
<th>IDWK Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_{11} )</td>
<td>0.19</td>
<td>0.18</td>
</tr>
<tr>
<td>( \beta_{21} )</td>
<td>0.53</td>
<td>0.55</td>
</tr>
<tr>
<td>( \beta_{31} )</td>
<td>0.10</td>
<td>0.09</td>
</tr>
<tr>
<td>( \beta_{41} )</td>
<td>0.19</td>
<td>0.18</td>
</tr>
</tbody>
</table>

Since this is an extension of the previous cases, as expected, estimates and actual regression weights are again in agreement.
g. Case 1420: Branched Reservoir – Three producers & Two Injector:

Producer P4 is replaced by injector I2 for this case. Distances and permeabilities are the same as for the previous case (Figure 2-11).

![Diagram of branched reservoir well locations]

**Figure 2-11** Case 1420. Branched reservoir well locations. Length and permeability values of the branches are given with respect to the I1-P1 branch. $l =$ length of I1-P1 branch; $k =$ permeability of I1-P1 branch.

Before applying equation (2.21), we proceed to calculate the equivalent permeability between injector I2 and producer P2 ($k_{22}$). Since the zones between I2-I1 and I1-P2 are in series, we apply harmonic averaging:
\[
\frac{1}{k_{22}} = \frac{1}{3} \left( \frac{1}{1} + \frac{2}{6} \right) = \frac{4}{9} \quad k_{22} = \frac{9}{4}
\]

**Table 2-10** Case 1420. IDWK calculations

<table>
<thead>
<tr>
<th></th>
<th>I1</th>
<th>I2</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>[\beta_{11} = \frac{1}{1 + \frac{6}{2} + \frac{1}{2}} \equiv 0.22]</td>
<td>[\beta_{12} = \frac{1}{\frac{9}{4} + \frac{1}{3} + \frac{1}{3}} \equiv 0.32]</td>
</tr>
<tr>
<td>P2</td>
<td>[\beta_{21} = \frac{6}{1 + \frac{6}{2} + \frac{1}{2}} \equiv 0.67]</td>
<td>[\beta_{22} = \frac{9}{4}{\frac{3}{2}} + \frac{1}{3} + \frac{1}{3} \equiv 0.47]</td>
</tr>
<tr>
<td>P3</td>
<td>[\beta_{31} = \frac{1}{\frac{6}{2} + \frac{1}{2} + \frac{1}{2}} \equiv 0.11]</td>
<td>[\beta_{32} = \frac{1}{\frac{9}{4} + \frac{1}{3} + \frac{1}{3}} \equiv 0.21]</td>
</tr>
</tbody>
</table>

**Table 2-11** Case 1420. IBMLR vs. IDWK weight estimate comparison

<table>
<thead>
<tr>
<th>IBMLR</th>
<th>IDWK Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\beta_{11})</td>
<td>0.23</td>
</tr>
<tr>
<td>(\beta_{12})</td>
<td>0.24</td>
</tr>
<tr>
<td>(\beta_{21})</td>
<td>0.64</td>
</tr>
<tr>
<td>(\beta_{22})</td>
<td>0.62</td>
</tr>
<tr>
<td>(\beta_{31})</td>
<td>0.12</td>
</tr>
<tr>
<td>(\beta_{32})</td>
<td>0.14</td>
</tr>
</tbody>
</table>

Two things can be observed from Table 2-11: first the IDWK estimate of the weights differ significantly from the actual weights for injector I02 (\(\beta_{12}, \beta_{22}, \beta_{32}\)) and second, I1 and I2 IBMLR weights are very similar for the same producer (\(\beta_{11} \equiv \beta_{12}, \beta_{21} \equiv \beta_{22}, \beta_{31} \equiv \beta_{32}\)).

Note that this injection data and reservoir configurations have been used previously with good results. However, Figure 2-11 shows that the model presents...
branching for injector I2 at the I1’s location. In fact, all flow from I02 is forced into the I1 location. I2 effectively acts as a second injector collocated with injector I1. Thus, we should expect both wells I1 and I2 to have similar weights; the properties of the reservoir branch connecting the injectors have no effect whatsoever other than affecting the pressures, which are not used in IBMLR.

To test this, we performed a sensitivity analysis to the permeability in the said branch in cases 1421 and 1422.
h. Cases 1421 and 1422 – Permeability Sensitivity in Dead End Reservoir

Permeability was set to one half and ten times respectively (Figure 2-12 and Figure 2-13).

![Diagram of cases 1421 and 1422 showing permeability sensitivity in a dead end reservoir.]

Figure 2-12 Case 1421. Permeability of branch I2-I1(underlined) reduced to one half.

The results, presented in Table 2-12, indicate that the weights indeed ignore the volume between I1 and I2. Remember that we are dealing with a small compressibility system and that this might not be true under higher compressibility or in a real 2D system.
Table 2-12 IBMLR weight sensitivity results to branch I2-I1 permeability.

<table>
<thead>
<tr>
<th>Case</th>
<th>Perm</th>
<th>1/2k</th>
<th>10k</th>
</tr>
</thead>
<tbody>
<tr>
<td>1420</td>
<td>β₁₁</td>
<td>0.23</td>
<td>0.23</td>
</tr>
<tr>
<td></td>
<td>β₁₂</td>
<td>0.24</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>β₂₁</td>
<td>0.64</td>
<td>0.65</td>
</tr>
<tr>
<td></td>
<td>β₂₂</td>
<td>0.62</td>
<td>0.63</td>
</tr>
<tr>
<td></td>
<td>β₃₁</td>
<td>0.12</td>
<td>0.13</td>
</tr>
<tr>
<td></td>
<td>β₃₂</td>
<td>0.14</td>
<td>0.15</td>
</tr>
</tbody>
</table>

**Figure 2-13** Case 1422. Permeability of branch I2-I1 (underlined) increased ten fold.
i. **Case 1423: Skeletonized Reservoir – Four Producers and Two Injectors:**

A new producer P4 along with a new branch for injector I2 is introduced in this model (Figure 2-14). The new branch has base properties (1L; 1K). For I2, this reservoir presents multiple branching; flow from the I2-I1 branch gets divided downstream of the I1 location into three (I1-P1, I1-P2, I1-P3).

![Diagram](insert_diagram)

**Figure 2-14** Case 1423 Branch P4-I2 has been to the branched reservoir used in case 1420.

The following observation can be made by inspecting Table 2-13
Since flow originating at I1 does not present branching downstream, I1 weights can be readily computed using IDWK (Table 2-14), which compare favorably with actual regression weights (Table 2-13):

**Table 2-13** IBMLR results for case 1423

<table>
<thead>
<tr>
<th></th>
<th>IBMLR weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{11}$</td>
<td>0.21</td>
</tr>
<tr>
<td>$\beta_{12}$</td>
<td>0.11</td>
</tr>
<tr>
<td>$\beta_{21}$</td>
<td>0.58</td>
</tr>
<tr>
<td>$\beta_{22}$</td>
<td>0.29</td>
</tr>
<tr>
<td>$\beta_{31}$</td>
<td>0.11</td>
</tr>
<tr>
<td>$\beta_{32}$</td>
<td>0.06</td>
</tr>
<tr>
<td>$\beta_{41}$</td>
<td>0.10</td>
</tr>
<tr>
<td>$\beta_{42}$</td>
<td>0.53</td>
</tr>
</tbody>
</table>

**Table 2-14** IDWK weights for injector I1

<table>
<thead>
<tr>
<th></th>
<th>IDWK Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{11}$</td>
<td>0.20</td>
</tr>
<tr>
<td>$\beta_{21}$</td>
<td>0.60</td>
</tr>
<tr>
<td>$\beta_{31}$</td>
<td>0.10</td>
</tr>
<tr>
<td>$\beta_{41}$</td>
<td>0.10</td>
</tr>
</tbody>
</table>

Based on observation of Table 2-13 and Table 2-14, the following relationship arises between the I2’s weights and I1’s weights:

$$\beta_{j2} = (1-\beta_{42})\beta_{j1} \quad j=1,2,3$$

(2.22)

The following results compare favorable with IBMLR results in Table 2-13:

$$\beta_{12} = (1-\beta_{42})\beta_{11} = (0.47)(0.23) = 0.11$$

$$\beta_{22} = (1-\beta_{42})\beta_{21} = (0.47)(0.58) = 0.27$$

$$\beta_{32} = (1-\beta_{42})\beta_{31} = (0.47)(0.11) = 0.05$$
That is to say, the sum of the weights for I2 that are subject to downstream branching \((1-\beta_{42})\), partition according to the weights of an injector located at the branching location (I1).
2.3.2. One-Dimensional Experiment Conclusions

Throughout these series of numerical experiments in a one-dimensional reservoir it has been observed that the physical estimates (IDW, IDWK) agree well with the IBMLR regression weights.

We have developed insights about how the weights should behave and what could be considered physically reasonable values. Within the context of this interpretation, we have confirmed that the weight values should lie between zero and one.

The interpretation of the permeability between an injector and a producer must be in an average sense. For linear models, harmonic averaging would, thus, be a reliable estimator of the average that would go into the physical estimate of the weights.

Although branching of flow at the injector as presented is unlikely to occur in two or three dimensions, the understanding of the behavior of the weights in such an environment contributes to the overall understanding of the physical behavior of the weights. In particular, injectors can be assimilated to injected water branching locations. If flow is concatenated through more than one branching location the physical problem becomes non-linear due to the arising product of weights.

Most importantly, we have learned that injectors contribute independently. Physically, the results should be analyzed one injector at a time.
In the next section, further testing will be performed in a different two-dimensional reservoir. For this we will make extensive use of the 5x4 Synfield and 25x16 Synfield that Albertoni originally used in his thesis.
2.3.3. Experiments in Two Dimensions

This set of experiments is aimed at testing the Eq. (2.7) relationship, namely the particular case of inverse distance weighting (IDW). Just as with the one-dimensional cases, we will be analyzing the output of synthetic 2D models.

a. Synfields

We will make extensive use of Albertoni’s synthetic fields in this section and subsequent chapters. A brief introduction to these is warranted at this point. See Albertoni (2002) for further reference.

![Figure 2-15 Albertoni’s 5x4 Synfield](image)

Fields of two sizes are studied: one of 5 injectors and 4 producers (the 5x4 Synfield) and one of 25 injectors and 16 producers (the 25x16 Synfield). Both are filled with an undersaturated oil and both have a five-spot injection pattern. The
system is initially at irreducible water saturation of 0.35. The injector-producer
distance is 800 ft for the 5x4 Synfield and 890 ft for the 25x16 Synfield. Unless
otherwise stated, horizontal permeability is 40md and vertical permeability 4md. The
oil-water mobility ratio is equal to one, and the oil, water, and rock compressibility
are 5.0, 1.0 and 1.0 mips, respectively. Both Synfields have vertical wells. A plan
view of Albertoni’s 5x6 Synfield is presented in Figure 2-15.

One important characteristic of this field that is particularly well suited for our
analysis is the small compressibility. Most transients are shorter than the monthly
sampling interval for the data.

\section*{b. Case 15: Homogeneous 5x4 Synfield}

For the 5x4 Synfield the inverse distance weighting results are summarized in
Table 2-15. These compare favorably with Albertoni’s results (Table 2-16).
Moreover, note that in terms of prediction, IBMLR and IDW weights result in similar
R² statistics. But, while IBMLR prediction shows a small bias, IDW prediction is
unbiased (Figure 2-16 and Figure 2-17). The latter observation suggests that the
IBMLR model should be revised to include zero bias constraints.

\begin{table}[h]
\centering
\caption{5x4 Homogeneous Synfield. IDW weights}
\begin{tabular}{|c|c|c|c|c|}
\hline
 & P1 & P2 & P3 & P4 \\
\hline I1 & 0.34 & 0.34 & 0.15 & 0.15 \\
I2 & 0.34 & 0.15 & 0.34 & 0.15 \\
I3 & 0.25 & 0.25 & 0.25 & 0.25 \\
I4 & 0.15 & 0.34 & 0.15 & 0.34 \\
I5 & 0.15 & 0.15 & 0.34 & 0.34 \\
\hline
\end{tabular}
\end{table}
Table 2-16 5x4 Homogeneous Synfield. IBMLR results (from Albertoni, 2002).

<table>
<thead>
<tr>
<th></th>
<th>P1</th>
<th>P2</th>
<th>P3</th>
<th>P4</th>
</tr>
</thead>
<tbody>
<tr>
<td>I1</td>
<td>0.33</td>
<td>0.33</td>
<td>0.17</td>
<td>0.17</td>
</tr>
<tr>
<td>I2</td>
<td>0.33</td>
<td>0.17</td>
<td>0.33</td>
<td>0.17</td>
</tr>
<tr>
<td>I3</td>
<td>0.24</td>
<td>0.25</td>
<td>0.25</td>
<td>0.26</td>
</tr>
<tr>
<td>I4</td>
<td>0.17</td>
<td>0.33</td>
<td>0.17</td>
<td>0.33</td>
</tr>
<tr>
<td>I5</td>
<td>0.17</td>
<td>0.16</td>
<td>0.34</td>
<td>0.33</td>
</tr>
</tbody>
</table>

Figure 2-16 IBMLR: Liquid Production Rate (Estimated vs Real).
c. **Case 44: Faulted Homogeneous 5x4 Synfield**

Starting from this case, we will adopt the realistic injection rates generated by Albertoni for input into the subsequent numerical reservoir models. The injection data was generated by randomly selecting different injectors in the Chihuido de la Sierra Negra Field, in Argentina, and renormalizing the rates to be in agreement with the Synfield injectivity (Figure 2-19).

For Albertoni’s faulted homogeneous 5x4 Synfield we find that the IBMLR and the inverse distance weighting again compare favorably (Table 2-17). However,
knowledge about the location of the fault is needed. This could be exploited to infer the correct location of a fault as described in Chapter 6 and in the following case.

Note that since injector-producer pairs across the fault from each other are presumed not to be in hydraulic communication, the values of the IDW for are not defined or alternatively could be defined as zero.

![5x4 Faulted Synfield. Fault is represented by the dashed line.](image)

**Figure 2-18** 5x4 Faulted Synfield. Fault is represented by the dashed line.
**Figure 2-19** Injection rates based real data set. rb/d = reservoir barrels/day, WVIR = injection rate.

**Table 2-17** Faulted 5x4 Synfield. Left: IBMLR results from Albertoni (2002). Right: IDW results. Well pairs across fault have no entries.

<table>
<thead>
<tr>
<th>IBMLR results (from Albertoni 2002)</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>P2</td>
<td>P3</td>
<td>P4</td>
<td></td>
</tr>
<tr>
<td>I1</td>
<td>-0.01</td>
<td>0.63</td>
<td>-0.02</td>
<td>0.36</td>
</tr>
<tr>
<td>I2</td>
<td>0.50</td>
<td>-0.01</td>
<td>0.50</td>
<td>0.00</td>
</tr>
<tr>
<td>I3</td>
<td>0.00</td>
<td>0.56</td>
<td>0.02</td>
<td>0.51</td>
</tr>
<tr>
<td>I4</td>
<td>-0.01</td>
<td>0.50</td>
<td>-0.03</td>
<td>0.51</td>
</tr>
<tr>
<td>I5</td>
<td>0.34</td>
<td>-0.01</td>
<td>0.73</td>
<td>-0.01</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Inverse Distance Weighting</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>P2</td>
<td>P3</td>
<td>P4</td>
<td></td>
</tr>
<tr>
<td>I1</td>
<td>-</td>
<td>0.69</td>
<td>-</td>
<td>0.31</td>
</tr>
<tr>
<td>I2</td>
<td>0.50</td>
<td>-</td>
<td>0.50</td>
<td>-</td>
</tr>
<tr>
<td>I3</td>
<td>-</td>
<td>0.50</td>
<td>-</td>
<td>0.50</td>
</tr>
<tr>
<td>I4</td>
<td>-</td>
<td>0.50</td>
<td>-</td>
<td>0.50</td>
</tr>
<tr>
<td>I5</td>
<td>0.31</td>
<td>-</td>
<td>0.69</td>
<td>-</td>
</tr>
</tbody>
</table>
**d. Case 500: Faulted 25x16 Synfield**

The 25x16 Synfield consists of 25 injectors and 16 producers arranged in five spot patterns (Figure 2-20). The injector–producer distance is 891 ft. All other model characteristics except fault locations are inherited from the 5x4 case.

*Figure 2-20* Faulted 25x16 Synfield. Well and fault locations.
The reservoir presents five compartments separated by four faults. Let’s assume that we suspect of their existence and would like to test whether they behave as flow barriers or not. If the barriers do exist and the compartments are isolated, then performing a regression on the whole field would be incorrect and viceversa.

Figure 2-21 Case 500. Left: results for regression on the whole field. IDW vs. IBMLR crossplot (upper left) shows large dispersion and negative weights. Regression results for producer P1 (lower left) shows the rate estimation to be biased. Right: results for regression within the compartments. A good agreement is achieved between IBMLR and IDW (upper right). The estimation of P1 rate are unbiased (lower right).
Figure 2-21 shows that IDW and IBMLR weights are in agreement when performing the estimations under the assumptions that the faults behave indeed as flow barriers (which is the actual case in the reservoir simulation model). Moreover, under this assumption, the estimated production rates show no bias and $R^2$ is slightly higher when assuming faults as barriers; 0.96 vs. 0.94.

Hence cross plotting IBMLR weights and IDW weights could prove an useful tool for assessing which configuration of producer-injection pairs works best for a given problem. In particular it can prove useful for testing the existence or not of flow barriers.

Performing the IBMLR on the whole fields results in a large number of negative weights; 26% (Figure 2-21, upper left). These cannot be explained by co-linearity or noise in the synthetic data alone. Thus, many negative weights might be an indication that the choice of injectors to regress against is not correct. Most of the negative weights are associated with pairs of producer-injectors across the sealing faults (Figure 2-22). Albertoni’s procedure of dropping the most negative weights successively is a valid approach. However, it may not necessarily result in consistent blocks or group of wells.

Nevertheless, a similar approach procedure is suggested to overcome this limitation. A strategy could be to map regression weights corresponding only to neighboring injector-producer pairs. In some cases this might be enough to identify blocks of wells (Figure 2-23, left). In other cases a cutoff value might be needed. This cutoff value might be suggested by IDW (ie: 50% of IDW for the corresponding pair). If the weight exceeds the cutoff value, then the connection is validated and vice
versa (Figure 2-23, right). Once well grouping is established, a second regression can be performed within groups to presumably obtain more robust results.

Figure 2-22 Case 500, faulted 25x16 Synfield. Map of IBMLR regression weights. Vectors originate at the injectors and are proportional to the weight value. Negative weights are plotted in blue. Red circles identify producers, blue circles identify injectors.
Figure 2-23 Case 500 25x16 faulted Synfield. Identifying blocks of wells. Left: IBMLR weights plotted for neighboring well pairs. Well grouping is suggested. Right: similar but a cutoff value for the weights is chosen. If the weight exceeds the cutoff, a connection is established. Once well grouping is established, regression can be performed within groups to presumably obtain more robust results.

This case also shows that performing the regression using all the data (injectors) available does not guarantee the minimum of the objective function. In this case, regressing on a subset of the available data, which is equivalent to forcing certain weights to be zero (arguably a more constrained case), resulted in a higher coefficient of determination $R^2$ and eliminated bias.

Finally Figure 2-21 suggests that if one no-bias constraint is applies to each producer, the resulting regression weights are bound to be more consistent with the underlying physical characteristics of the reservoir.

$$E(\hat{P}_j) = E(P_j) \quad \text{for all } j$$  \hspace{1cm} (2.23)
As mentioned before, Albertoni observed that for a homogeneous field, the weights group according to the field's planes of symmetry. He used a symmetry coefficient to assess the quality of the regression weights. He established that these planes of symmetry determined three groups of injector-producer pairs with similar relative location; and that these groups for the 5x4 Synfield are: (a) corner injectors with adjacent producers I1-P1, I1-P2, I2-P1, I2-P3, I4-P2, I4-P4, I5-P3, and I5-P4; (b) corner injectors with non-adjacent producers I1-P3, I1-P4, I2-P2, I2-P4, I4-P1, I4-P3, I5-P1, and I5-P2; and (c) center injector with adjacent producers I3-P1, I3-P2, I3-P3, and I3-P4.

For the 5x4 Synfield these symmetries compare favorable to the three combinations of weights calculated. Moreover we can determine the values of groups a, b, c to be 0.34, 0.15, and 0.25 correspondingly.

*Table 2-18.* 5x4 Synfield. Homogeneous reservoir. Albertoni’s well-pair groups determined by planes of symmetry (left) and inverse distance calculation (right).

<table>
<thead>
<tr>
<th>Symmetry Groups</th>
<th>P1</th>
<th>P2</th>
<th>P3</th>
<th>P4</th>
</tr>
</thead>
<tbody>
<tr>
<td>I1</td>
<td>a</td>
<td>a</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>I2</td>
<td>a</td>
<td>b</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>I3</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
</tr>
<tr>
<td>I4</td>
<td>b</td>
<td>a</td>
<td>b</td>
<td>a</td>
</tr>
<tr>
<td>I5</td>
<td>b</td>
<td>b</td>
<td>a</td>
<td>a</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Inverse Distance Weighting</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
</tr>
<tr>
<td>----</td>
</tr>
<tr>
<td>I1</td>
</tr>
<tr>
<td>I2</td>
</tr>
<tr>
<td>I3</td>
</tr>
<tr>
<td>I4</td>
</tr>
<tr>
<td>I5</td>
</tr>
</tbody>
</table>

Albertoni further that suggests $R^2$ alone is not a good indicator of the quality of the correlation between observed and modeled production. and that weight symmetry is a much better indicator and should be considered together with $R^2$ to
evaluate the goodness of the results. On the other hand, he further argues that in actual applications of the technique one must only use $R^2$ because symmetry is not expected from the results because symmetry arises from the symmetric well locations in the Synfields and the fact that the field is homogeneous according to IDW (2.8). In actual applications, the homogeneity assumption would not hold and thus, as mentioned, symmetry would not be expected.
2.4. NUMERICAL EXPERIMENTS II: TESTING SUPERPOSITION;
INDIVIDUAL RATE CONTRIBUTION FROM INJECTORS

This set of numerical experiments is aimed at testing relationship Eq. (2.5) in a more straightforward way.

\[
\beta_{ij} = \frac{q_{ij}}{I_i}
\]  

(2.5)

As mentioned before, described in this manner, the \( \beta \) weight is the ratio of the rate between any injector-producer pair to the total injection rate. Note that in the derivation of this relationship, there was no reference to the presence of any other injector. The effect of other injectors is to change the flow configuration or streamline paths. The producers, on the other hand, act as boundary conditions (constant pressure nodes) and their number, location and operating conditions affect the weights.

The method will be as follows. First \( \beta \) weights are obtained from regression using a production history dataset generated from a Synfield as shown in Figure 2-24. These will be the reference weight against which we wish to test expression Eq. (2.5).

![Figure 2-24 Schematic showing the method for obtaining the reference weights.](image)

67
To obtain the weights as described in Eq. (2.5), we apply the following outlined procedure:

1. Select the first injector \( I_1 \).

2. Run the Synfield excluding all injectors but the selected one from the model. A production-injection data set is obtained for this injector.

3. Using the generated dataset, after the initial primary production period decays, compute the ratio of the production rate to injector rate for each producer at each time period. The volumes produced by each producer, under these conditions, can only be sustained by the volumes flowing from the injector to each of the producers. Thus, production rates \( P_j \) would be equivalent to \( q_{ij} \) for the particular injector \( I_1 \) selected.

4. Average the \( P_j/I_1 \) ratios to obtain \( \beta_{j1} \) according to Eq. (2.5)

5. Select next injector and repeat the procedure to obtain the whole set of \( \beta_{ji} \) weights (Figure 2-25).

\[
\begin{align*}
\beta_{j1} &= E \left( \frac{q_{j1}}{I_1} \right) \\
\beta_{j2} &= E \left( \frac{q_{j2}}{I_2} \right) \\
\beta_{j3} &= E \left( \frac{q_{j3}}{I_3} \right) \\
\beta_{j4} &= E \left( \frac{q_{j4}}{I_4} \right) \\
\beta_{j5} &= E \left( \frac{q_{j5}}{I_5} \right)
\end{align*}
\]

Figure 2-25 For the 5x4 field, five runs are needed; one for each injector. From each run, a set of weights corresponding to a particular injector is obtained. From left to right, the well configurations for each run are presented.
2.4.1. 5x4 Homogeneous Synfield

We already know what the values weights for this field should be. We have previously applied IBMLR to obtain the values in Table 2-19 (right) and have obtained the same results using inverse distance weighting physical approximation.

We now apply the procedure described above to this same field obtaining the single injector allocation shown on Table 2-18 (left) and summarized in Figure 2-26. The agreement between both estimates is remarkable. Note that for Injector I03, we should obtain the same weights (0.25) for all producers and a correlation calculation is, thus, meaningless.

Table 2-19 Case 2000. 5x4 Homogeneous Synfield Single Well Allocation (left) vs. IBMLR weights (right).

<table>
<thead>
<tr>
<th>Single Injector Well Allocation</th>
<th>IBMLR Weights:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>P1</td>
</tr>
<tr>
<td>11</td>
<td>0.329</td>
</tr>
<tr>
<td>12</td>
<td>0.330</td>
</tr>
<tr>
<td>13</td>
<td>0.250</td>
</tr>
<tr>
<td>14</td>
<td>0.171</td>
</tr>
<tr>
<td>15</td>
<td>0.171</td>
</tr>
</tbody>
</table>
Figure 2-26 Case 2000. 5x4 Homogeneous Synfield Single Injector Allocation vs. IBMLR weights.

2.4.2. 5x4 Heterogeneous Synfield

The same analysis is now conducted in the Albertoni’s heterogeneous 5x4 Synfield. There is now a high permeability band that spans the field from east to west encompassing the northern most fourth of the field. The northern zone has ten times the permeability of the southern zone, which is 40 md.
The weights, in this case, span most of the physical range (0.05-0.8). Again, the results show very good agreement between the IBMLR weights and the single injector allocation scenarios (Figure 2-28).

Thus the IBMLR weights also represent the fraction of water injected flowing between an injector and a producer if this injector was the only one present in the field.
Figure 2-28 Case 2100. 5x4 Heterogeneous Synfield Single Well Allocation vs. IBMLR weights.

### 2.4.3. 25x16 Homogeneous Synfield

This case shows more dispersion around the forty-five degree line but the trend is clear nonetheless. Notice also that there are a few negative weights occurring at small weights which, can be associated with the more distant wells. These negative weights cause overshooting of positive weights corresponding to the same injector.
This weight behavior suggests that, as expected, if the field is large and distant injectors are used, eventually we get negative weights because the signals between far away pairs are too weak to be used in a robust manner. From a pragmatic point of view, this in turn suggests that a distance cutoff as suggested by Albertoni might be useful. Alternatively, excluding these injectors from the regression and/or constraining the weights to be positive might be a better approach.
2.4.4. Discussion and Conclusions

Recall that while the values from the IBMLR are regressed from rates in which all wells are acting collectively, the values for the single injector well allocation are computed from simulation runs that have one injector at a time. This would suggest that the linear model expresses the production rate as a superposition of independent contributions from each of the injectors.

The rate between each of the producer-injector pairs is the result of the pressure field arising from injecting at the injector location given certain permeability field and boundary conditions. When more than one injector is present, a composite pressure field is generated and the actual flow geometry responds accordingly. Accordingly we do not expect the $q_{ij}$ rates to represent actual flow rates between an injector-producer pair. The $q_{ij}$ rates represent in term of rates the contribution of the injectors to the pressure field. Furthermore, this contribution is an average that represents the behavior within a volume between the injector and the producer. In other words, the $q_{ij}$ rates represent the transmissibility-weighted pressure difference between an injector and a producer.

The weights, $\beta_{ij}$, represent the normalization of the $q_{ij}$ rates with respect to the total injection rate or to the total transmissibility-weighted gradients a given injector generates.

Negative weights have no physical explanation. In this section we have come across them as a result of misidentifying hydraulically independent reservoir blocks as a single hydraulic unit.
Other sources of negative weights are co-linearity in the dataset (Yousef, in progress) or high compressibility in the reservoir. When significant negative weights are present, quantitative use of the weights is not recommended.

Results from section 2.3.3.d suggest that the IBMLR might benefit from additional non-bias constraints for each producer. This might reduce the occurrence of significant negative weights across hydraulically independent blocks.
2.5. NUMERICAL EXPERIMENTS III: INJECTOR SHUT-IN ANALYSIS

Previous results suggest that the injector’s particular behavior should not have an impact on the arising weights (Albertoni, 2002). Unless a shut in period creates an artifact in the dataset (such as co-linearity), there should be no reason for it not to be handled properly by the model, moreover, it should add to the injector’s signal signature.

In this section we explore this issue by analyzing the impact on the IBMLR weights of introducing arbitrary shut in periods on datasets analyzed in previous sections.

2.5.1. Case 1041: 4x4 Synfield; inactive injector

This case is based on the homogeneous 5x4 Synfield. Injector I3 has been removed from the model to represent the extreme case of a shut in that lasts the whole analysis time window (Figure 2-30). From the previous results, we expect the weights to be unaffected.

![Figure 2-30 Case 1041 4x4 Homogeneous Synfield](image-url)
This is indeed the case as can be observed in Figure 2-31 and Table 2-20.

<table>
<thead>
<tr>
<th>Case 1041 5x4 Homogeneous Base Case weights vs. 4x4 Homogeneous Case.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>IBMLR Weights - No Shut In</strong></td>
</tr>
<tr>
<td>I01</td>
</tr>
<tr>
<td>I02</td>
</tr>
<tr>
<td>I03</td>
</tr>
<tr>
<td>I04</td>
</tr>
<tr>
<td>I05</td>
</tr>
</tbody>
</table>

**Figure 2-31** Case1041 5x4 Base Case weights vs. 4x4 Case.

2.5.2. **Case 1042: 5x4 Synfield, Multiple shut-ins in one injector.**

Two three-month shut-in periods separated by five months of production have been chosen as a representative situation for multiple shut-in periods for one well (Figure 2-32).

Figure 2-33 and Table 2-21 show that the IBMLR solution gives the same results whether the injector is shut in or not; moreover, these results are also in
agreement with the IDW estimator (Table 2-15). This is clearly not the case for the MLR and ABMLR systems as depicted in Table 2-22 and Figure 2-34.

![Injection rate data](image)

**Figure 2-32** Injection rate data for case 1042. Injector I03 is shut in twice.

**Table 2-21** Comparison between a homogeneous case with no shut in periods and the same case with two shut-in periods for the same well.

<table>
<thead>
<tr>
<th></th>
<th>IBMLR Weights - No Shut In</th>
<th>IBMLR Weights - I03 shuts in twice</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>P01  P02  P03  P04  Lagr Mult</td>
<td>P01  P02  P03  P04  Lagr Mult</td>
</tr>
<tr>
<td>I01</td>
<td>0.330  0.328  0.172  0.170  16.28</td>
<td>0.330  0.327  0.172  0.170  20.56</td>
</tr>
<tr>
<td>I02</td>
<td>0.326  0.173  0.327  0.174  -78.69</td>
<td>0.326  0.173  0.327  0.174  -154.5</td>
</tr>
<tr>
<td>I03</td>
<td>0.249  0.251  0.249  0.251  -158.7</td>
<td>0.250  0.250  0.250  0.250  -3605</td>
</tr>
<tr>
<td>I04</td>
<td>0.175  0.331  0.169  0.325  -199.8</td>
<td>0.175  0.331  0.169  0.325  7.379</td>
</tr>
<tr>
<td>I05</td>
<td>0.166  0.179  0.321  0.335  87.71</td>
<td>0.166  0.179  0.321  0.334  -226.4</td>
</tr>
</tbody>
</table>
Figure 2-33 Comparison between a homogeneous case with no shut in periods and the same case with two shut-in periods for the same well.

Table 2-22 MLR and ABMLR Regression results for case 1042.

<table>
<thead>
<tr>
<th>ABMLR Weights - I03 shut in twice</th>
<th>MLR Weights - I03 shut-in twice</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>P2</td>
</tr>
<tr>
<td>I1</td>
<td>0.427</td>
</tr>
<tr>
<td>I2</td>
<td>0.358</td>
</tr>
<tr>
<td>I3</td>
<td>0.212</td>
</tr>
<tr>
<td>I4</td>
<td>0.118</td>
</tr>
<tr>
<td>I5</td>
<td>0.121</td>
</tr>
</tbody>
</table>

Table 2-23 Absolute deviation of weight from the IBMLR values. Case 1042

<table>
<thead>
<tr>
<th>IBMLR-ABMLR Weights - I03 shut in twice</th>
<th>IBMLR-MLR Weights - I03 shut in twice</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>P2</td>
</tr>
<tr>
<td>I1</td>
<td>0.097</td>
</tr>
<tr>
<td>I2</td>
<td>0.032</td>
</tr>
<tr>
<td>I3</td>
<td>-0.037</td>
</tr>
<tr>
<td>I4</td>
<td>-0.058</td>
</tr>
<tr>
<td>I5</td>
<td>-0.045</td>
</tr>
</tbody>
</table>
Notice on Table 2-23 that the deviation of the ABMLR and MLR weights from the IBMLR values is fairly constant for a given injector. Moreover, the sum of these deviations for a given injector compare favorably to the sum of the weights for a given injector (Table 2-24) minus one as described by equations (1.59) and (1.61).

These results are in agreement with the relationship found between the weights of the three systems in section 1.2.5.
2.6. CONCLUSIONS

The IBMLR system of equations represents a physical system in which injectors independently contribute a fraction of their rate to the producers. Under this context, the weights can be interpreted as the fraction of injected water that would flow from an injector to a producer if all other injectors were shut-in. This is also equivalent to the relative average transmissibility between an injector-producer pair with respect to the sum of transmissibilities between said injector and all producers. This has been tested in 1D, 2D, homogeneous and heterogeneous reservoirs.

The system is able to deal with injector shut-in periods as long as the shut-in periods of different injectors do not overlap. This would cause artifacts such as colinearity (Yousef, in progress).
CHAPTER 3: GENERIC MODEL – INCORPORATION OF
BOTTOMHOLE PRESSURE CHANGE

We have already observed that the instant balanced multilinear regression (IBMLR) system Eq. (2.4) is adequate when the flowing pressures are the same for all producers. The “injection from production” (IFP) Eq. (2.14) system would be more adequate when the reverse condition is true; that is, the flowing pressure for the injectors is the same with fluctuations both in rate and flowing pressures on the producers.

In this chapter we introduce bottomhole pressure information into the model overcoming one of its main limitations. It has been observed that varying bottomhole pressures have an impact on the weights (Albertoni 2002, Thang unpublished). This observation is confirmed by the physical estimator Eq. (2.7) derived previously. Incorporating bottomhole pressure information into the model will not only result in more robust regression weights but also in an additional set of producer-producer interaction weights that carry information about the inter-producer region.

3.1. MODEL DESCRIPTION

We start by proposing the following extension to the MLR model where the fluctuating bottomhole pressures are associated with the production rate estimate by a second set of weights $\gamma$. 
\[ \hat{P}_j = \sum_i \beta_{ji} I_i + \sum_{k} N_{jvar} \gamma_{jk} BHP_k \]  

(3.1)

where:

- \( N_{jvar} \) is the number of producers with fluctuating bottomhole pressures
- \( BHP_k \) is the fluctuating bottomhole pressure corresponding to the \( k \)th producer
- \( i = 1 \ldots I \) is the index of the injectors

Note that the weights \( \gamma_{jk} \) have units; in this work RB/psi.

We also require the system to be balanced:

\[ \sum_i I_i = \sum_j P_j \]  

(3.2)

This results in a set of constraints for the weights:

\[ \sum_i I_i = \sum_j \left( \sum_i N_i \beta_{ji} I_i + \sum_{k} N_{jvar} \gamma_{jk} BHP_k \right) \]  

(3.3)

\[ \sum_i I_i = \sum_j \left( \sum_i \beta_{ji} I_i \right) + \sum_{k} N_{jvar} \left( \sum_j \gamma_{jk} \right) BHP_k \]  

(3.4)

Equation (3.4) is satisfied when

\[ \sum_j \beta_{ji} = 1 \]  

(3.5)

and

\[ \sum_j \gamma_{jk} = 0 \]  

(3.6)

The model and associated constrains are thus:

\[ \begin{align*}
\hat{P}_j &= \sum_i \beta_{ji} I_i + \sum_{k} N_{jvar} \gamma_{jk} BHP_k \\
\sum_j \beta_{ji} &= 1 \\
\sum_j \gamma_{jk} &= 0
\end{align*} \]  

(3.7)
To solve the system we minimize the objective function, which consists of the variance of the estimate error and the Lagrange multiplier constraint terms:

\[
\text{Min } \left[ \sum_j \text{Var}(\hat{p}_j - p_j) + \sum_i 2\mu_{\beta_i} \left( 1 - \sum_j \beta_{ji} \right) + \sum_k 2\mu_{\gamma_k} \left( \sum_j \gamma_{jk} \right) \right]
\]

Setting the derivatives of the weights and of the Lagrange multipliers equal to zero results in the following system of equations:

\[
\begin{pmatrix}
K & 0 & \cdots & 1 \\
0 & K & 0 & 1 \\
\vdots & 0 & \ddots & \vdots \\
1 & I & \cdots & 0 \\
\end{pmatrix}
\begin{pmatrix}
\beta \\
\gamma \\
\mu_\beta \\
\mu_\gamma \\
\end{pmatrix}
= 
\begin{pmatrix}
C_{\text{IP}} \\
C_{\text{BHP-P}} \\
C_{\text{I-BHP}} \\
C_{\text{BHP-BHP}} \\
\end{pmatrix}
\]

(3.8)

where

\[
K = \begin{pmatrix}
C_{\text{II}} & C_{\text{I-BHP}} \\
C_{\text{I-BHP}}^T & C_{\text{BHP-BHP}}
\end{pmatrix}
\]

Dimensions of \( K \) are \((I+N_{\text{jvar}}) \times (I+N_{\text{jvar}})\)

The number of times \( K \) is repeated along the diagonal of the left term covariance matrix in Eq. (3.8) is equal to the number of producers.

\( C_{\text{II}} \) is the injector-injector- covariance matrix with dimensions \( I \times I \)

\( C_{\text{I-BHP}} \) is the injector rate-producer bottomhole pressure covariance matrix with dimensions \( I \times N_{\text{jvar}} \).

\( C_{\text{BHP-BHP}} \) is the producer bottomhole pressure – producer bottomhole pressure covariance matrix with dimensions \( N_{\text{jvar}} \times N_{\text{jvar}} \).

\( C_{\text{I-BHP}}^T \) is the transpose of \( C_{\text{I-BHP}} \).

\( 0 \) is, for the left side term of Eq. (3.8), a matrix of the same dimensions as \( K \) containing all zeroes. For the right side of Eq. (3.8) \( 0 \) is a \( N_{\text{jvar}} \) long vector containing zeroes.
\( \textbf{I} \) is an I long vector containing ones.

\( \textbf{I} \) is an identity matrix of the same dimensions as \( \textbf{K} \).

\( \textbf{\beta} \) is a vector containing IBMLR-like weights with I\( \times \)N number of elements.

\( \gamma \) is a N\( \times \)N\text{\textunderscore}var long vector containing producer-producer interaction weights.

\( \mu_{\beta} \) is a vector containing the Lagrange multipliers that ensure constraint Eq. (3.5) for each injector. Vector length: I

\( \mu_{\gamma} \) is a vector containing the Lagrange multipliers that ensure constraint Eq. (3.6) for each of the producers whose bottomhole pressure is not constant. Vector length: N\text{\textunderscore}var.

\( \textbf{C}_{\text{IP}} \) is an I\( \times \)N long injection-production covariance vector.

\( \textbf{C}_{\text{BHP-P}} \) is a N\( \times \)N\text{\textunderscore}var long bottomhole pressure–production rate covariance vector.

On a side note, notice that system Eq. (3.7) resembles an ordinary cokriging system.

3.2. NUMERICAL EXPERIMENTS

We will proceed to test the generic model Eq. (3.7) using the cases presented in chapter 2 with operating conditions of the producers modified appropriately. One-dimensional cases are presented first followed by two-dimensional cases.

3.2.1. One Dimensional Cases

For cases 1211, 1211-1, 1211-2, and 1212-1 the fluctuating bottomhole pressure shown in Figure 3-1 was used for producer P1. The BHP for P1 equal to a
constant value plus a fluctuating term obtained from a uniform distribution between 0 and 200. This pressure is changed every two months (sampling interval). Bottomhole pressure for producer P2 remained unchanged at 250 psia.

![Fluctuating bottomhole pressure for producer P1.](image)

**Figure 3-1** Fluctuating bottomhole pressure for producer P1.

**a. Case 1211: Homogeneous reservoir, Equidistant Well Locations**

The reservoir, well location and injection rates for this case are the same as in case 1201 (Figure 3-2).

![Reservoir, well location are the same as for case 1201.](image)

**Figure 3-2** Case 1211. Reservoir, well location are the same as for case 1201.

On Figure 3-3 a strong correlation with BHP (Figure 3-1) can be observed for producer P2 as can an equally strong anti-correlation for producer P1. In Table 3-1 we can observe a strong departure from the values of the weights corresponding to the constant BHP case ($\beta_{11} = 0.5, \beta_{21} = 0.5$) when applying the IBMLR system while
ignoring the BHP data \((\beta_{11} = 0.33, \beta_{21} = 0.67)\). When BHP information is incorporated and system (3.7) (IBMLR+P) is used, the \(\beta_{ij}\) weights are restored to within 0.01 of their expected IDW values and agree with the constant and common BHP case.

Also, the \(\gamma_{jk}\) weights are of equal magnitude and opposite sign. Although this outcome is expected as a consequence of the constraint on the \(\gamma_{jk}\) weights Eq. (3.6), it is a desirable outcome that also agrees with rate behavior observed in Figure 3-3, which suggests that what one well gains the other loses, and vice-versa.

**Figure 3-3** Production rates for producers P1 and P2. The strong component of the BHP fluctuations and the resulting anti-correlation between P1 and P2 is evident.

**Table 3-1** Comparison of IBMLR regression results without BHP fluctuations (case 1201) and with fluctuating BHP (case 1211). Incorporating the BHP information (IBMLR+P) restores the injector-producer weights \((\beta_{ji})\).

<table>
<thead>
<tr>
<th></th>
<th>IDW</th>
<th>IBMLR (Case 1201)</th>
<th>IBMLR (Case 1211)</th>
<th>IBMLR+P (Case 1211)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\beta_{11})</td>
<td>0.50</td>
<td>0.50</td>
<td>0.33</td>
<td>0.51</td>
</tr>
<tr>
<td>(\beta_{21})</td>
<td>0.50</td>
<td>0.50</td>
<td>0.67</td>
<td>0.49</td>
</tr>
<tr>
<td>(\mu_1)</td>
<td>-</td>
<td>1.20</td>
<td></td>
<td>\mu_1</td>
</tr>
<tr>
<td>(\gamma_{11})</td>
<td>-0.40</td>
<td></td>
<td></td>
<td>\gamma_{11}</td>
</tr>
<tr>
<td>(\gamma_{21})</td>
<td>0.40</td>
<td></td>
<td></td>
<td>\gamma_{21}</td>
</tr>
<tr>
<td>(\mu_p)</td>
<td>-37.06</td>
<td></td>
<td></td>
<td>\mu_p</td>
</tr>
</tbody>
</table>
b. Case 1211-1: Homogeneous Case, Asymmetric Well Locations

This case is similar to constant bottomhole case 1201-1. The injector is moved off center to create asymmetry in the problem.

Similarly to the previous case, if the BHP information is ignored the resulting weights would not agree with the physical IDW interpretation, whereas when this data is incorporated through the IBMLR+P system not only are the $\beta_{ji}$ weights approximated by the IDW estimates, but the producer-producer weights $\gamma_{jk}$ that are obtained have the same values as in the previous (1211) case (Table 3-2). The latter implies that the $\gamma_{jk}$ weights are independent of the injector location.

Table 3-2 Comparison of IBMLR regression results without BHP fluctuations (case 1201-1) and with fluctuating BHP (case 1211-1). Incorporating the BHP information (IBMLR+P) restores the injector-producer weights ($\beta_{ji}$). $\gamma$ weights are the same as in the previous case.

<table>
<thead>
<tr>
<th></th>
<th>IDW</th>
<th>IBMLR (Case 1201-1)</th>
<th>IBMLR (Case 1211-1)</th>
<th>IBMLR+P (Case 1211-1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{11}$</td>
<td>0.70</td>
<td>0.69</td>
<td>0.52</td>
<td>$\beta_{11}$</td>
</tr>
<tr>
<td>$\beta_{21}$</td>
<td>0.30</td>
<td>0.31</td>
<td>0.48</td>
<td>$\beta_{21}$</td>
</tr>
<tr>
<td>$\mu_1$</td>
<td>-</td>
<td>1.15</td>
<td></td>
<td>$\mu_1$</td>
</tr>
<tr>
<td>$\gamma_{11}$</td>
<td></td>
<td></td>
<td></td>
<td>$\gamma_{11}$</td>
</tr>
<tr>
<td>$\gamma_{21}$</td>
<td></td>
<td></td>
<td></td>
<td>$\gamma_{21}$</td>
</tr>
<tr>
<td>$\mu_p$</td>
<td></td>
<td></td>
<td></td>
<td>$\mu_p$</td>
</tr>
</tbody>
</table>
c. **Case 1211-2: Heterogeneous 1D numerical model**

Just as in case 1201-2, the permeability of the right branch (locations 10-31) is increased 2.3 times (Figure 3-5).

![Figure 3-5 Case 1211-2. Two producers and one injector in an asymmetric and heterogeneous configuration.](image)

**Table 3-3** Comparison of IBMLR regression results without BHP fluctuations (case 1201-2) and with fluctuating BHP (case 1211-2). Incorporating the BHP information (IBMLR+P) restores the injector-producer weights ($\beta_{ji}$). $\gamma$ weights change is proportional to the average change in permeability between the producers.

<table>
<thead>
<tr>
<th></th>
<th>IBMLR (Case1201-2)</th>
<th>IDWK</th>
<th>IBMLR (Case1211-2)</th>
<th>IBMLR+P</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{11}$</td>
<td>0.50</td>
<td>0.50</td>
<td>0.21</td>
<td>$\beta_{11}$ 0.51</td>
</tr>
<tr>
<td>$\beta_{21}$</td>
<td>0.50</td>
<td>0.50</td>
<td>0.79</td>
<td>$\beta_{21}$ 0.49</td>
</tr>
<tr>
<td>$\mu_1$</td>
<td>0.99</td>
<td>0.99</td>
<td>$\gamma_{11}$</td>
<td>-0.67</td>
</tr>
<tr>
<td>$\gamma_{21}$</td>
<td></td>
<td></td>
<td>$\gamma_{21}$</td>
<td>0.67</td>
</tr>
<tr>
<td>$\mu_p$</td>
<td></td>
<td></td>
<td>$\mu_p$</td>
<td>-29.57</td>
</tr>
</tbody>
</table>

Again, the use of IBMLR+P results in weights similar to the IDW estimates. Also note that the change in absolute value of the producer-producer $\gamma_{jk}$ weights is proportional to the change in harmonic average permeability between both producers.

\[
k_{1211-1} = 1
\]

\[
k_{1211-2} = \frac{30}{8.5/1 + 21.5/2.3} = 1.68 \quad \text{(total length= 30 units)}
\]

\[
k_{1211-2} / k_{1211-1} = 1.68
\]

\[
\gamma_{1211-2} / \gamma_{1211-1} = \frac{0.67}{0.4} = 1.68
\]
d. Case 1212-1: Two producers, two injectors

In this case, two producers and two injectors in symmetric configuration are tested. Again, the BHP is fluctuating for producer P1 located in cell 1.

![Figure 3-6](image)

From Table 3-4 two observations can be made. First, the producer-producer weights $\gamma_{jk}$ return to the values observed for the previous homogeneous cases. Second, as in the previous cases, IBMLR+P injector-producer weights tend to the IDW estimates.

<table>
<thead>
<tr>
<th></th>
<th>IDW</th>
<th>IBMLR (Case 1212-1)</th>
<th>IBMLR+P (Case 1212-1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{11}$</td>
<td>0.71</td>
<td>0.54</td>
<td>$\beta_{11}$</td>
</tr>
<tr>
<td>$\beta_{12}$</td>
<td>0.29</td>
<td>0.33</td>
<td>$\beta_{12}$</td>
</tr>
<tr>
<td>$\beta_{21}$</td>
<td>0.29</td>
<td>0.46</td>
<td>$\beta_{21}$</td>
</tr>
<tr>
<td>$\beta_{22}$</td>
<td>0.71</td>
<td>0.67</td>
<td>$\beta_{22}$</td>
</tr>
<tr>
<td>$\mu_1$</td>
<td>-</td>
<td>-0.53</td>
<td>$\mu_1$</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>-</td>
<td>-29.90</td>
<td>$\mu_2$</td>
</tr>
</tbody>
</table>

From Table 3-4 two observations can be made. First, the producer-producer weights $\gamma_{jk}$ return to the values observed for the previous homogeneous cases. Second, as in the previous cases, IBMLR+P injector-producer weights tend to the IDW estimates.
e. Case 1216-2: Heterogeneous reservoir. More than one fluctuating BHP.

For this case the well configuration is the same as for case 1201-2. However, the BHPs of both producers fluctuate over time (Figure 3-8).

![Figure 3-7 Case 1216-2. Reservoir, well location are the same as for case 1201-2.](image)

![Figure 3-8 Fluctuating bottomhole pressures for producers P1 and P2. Gaps in dataset correspond to periods where the BHP was higher than the reservoir pressure.](image)

Two more terms need to be evaluated in the IBMLR+P model to account for the presence of a second BHP fluctuation. It can be observed (Table 3-5) that IBMLR+P successfully retrieves injector-producer weights that match IDW estimates. Producer-producer weights values are in agreement with those of case1211-2.
Figure 3-9 Production rates for case 1216-2. Gaps in dataset correspond to periods where the BHP was higher than the reservoir pressure.

Table 3-5 Case 1216-2 IBMLR+P results agree with 1211-2 IBMLR+P results and 1201-2 IBMLR results.

<table>
<thead>
<tr>
<th></th>
<th>IDW</th>
<th>IBMLR (Case 1201-2)</th>
<th>IBMLR (Case 1216-2)</th>
<th>IBMLR+P (Case1216-2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\beta_{11})</td>
<td>0.50</td>
<td>0.50</td>
<td>0.25</td>
<td>(\beta_{11})</td>
</tr>
<tr>
<td>(\beta_{21})</td>
<td>0.50</td>
<td>0.50</td>
<td>0.75</td>
<td>(\beta_{21})</td>
</tr>
<tr>
<td>(\mu_1)</td>
<td>-</td>
<td>3.93</td>
<td>-18.31</td>
<td>(\mu_1)</td>
</tr>
<tr>
<td>(\gamma_{11})</td>
<td></td>
<td></td>
<td>-0.65</td>
<td>(\gamma_{11})</td>
</tr>
<tr>
<td>(\gamma_{12})</td>
<td></td>
<td></td>
<td>0.64</td>
<td>(\gamma_{12})</td>
</tr>
<tr>
<td>(\gamma_{21})</td>
<td></td>
<td></td>
<td>0.65</td>
<td>(\gamma_{21})</td>
</tr>
<tr>
<td>(\gamma_{22})</td>
<td></td>
<td></td>
<td>-0.64</td>
<td>(\gamma_{22})</td>
</tr>
<tr>
<td>(\mu_{p1})</td>
<td></td>
<td></td>
<td>-18.46</td>
<td>(\mu_{p1})</td>
</tr>
<tr>
<td>(\mu_{p2})</td>
<td></td>
<td></td>
<td>31.80</td>
<td>(\mu_{p2})</td>
</tr>
</tbody>
</table>
3.2.2. Discussion of Partial Results for One Dimensional Cases

So far, these previous results indicate that:

- Incorporating the producer’s bottomhole pressure successfully corrects the deviation of the $\beta_{ji}$ weights from their physical IDWK estimates.
- Producer-producer $\gamma_{jk}$ weights are independent of injector location.
- Producer-producer $\gamma_{jk}$ weights scale with the average permeability of the volume between the producers.

Hence, the generic IBMLR+P model, so far, proves useful to overcome the constant BHP limitation of IBMLR. Moreover, the producer-producer weight behavior suggests that they might also contain useful information about the properties of the reservoir. A discussion of the physical meaning of the $\gamma_{jk}$ weights is thus warranted at this point.

3.3. Physical Meaning of the Producer-Producer Weights

Recall the system of equations for the IBMLR+P system:

$$\hat{p}_j = \sum_i \beta_{ji} I_i + \sum_{k} \gamma_{jk} BHP_k$$

$$\sum_j \beta_{ji} = 1$$

$$\sum_j \gamma_{jk} = 0$$

(3.7)

If we visualize the $\gamma_{jk} BHP_k$ terms as rate contributions towards $\hat{p}_j$ or as rate unbalances between the producers caused by difference in their flowing bottomhole pressures and if in addition we consider that the $\gamma_{jk} BHP_k$ term must have rate units, then we can postulate:
\( q_{jk} = \gamma_{jk} \text{BHP}_k \) (3.9)

where \( q_{jk} \) represents a flow rate between a pair of producers \( jk \).

As for the IBMLR weight interpretation we can make use of Darcy’s law for steady-state, one phase, linear system:

\[
q_{jk} = \frac{A_{jk} k_{jk} \Delta P_{jk}}{\mu L_{jk}} \tag{3.10}
\]

Equating (3.9) and (3.10):

\[
\gamma_{jk} \text{BHP}_k = \frac{A_{jk} k_{jk} \Delta P_{jk}}{\mu L_{jk}} \tag{3.11}
\]

At this point, since we are assuming the additional terms to represent flow between producers, it might be justifiable to revise the model. Instead of an expression in terms of \( \text{BHP}_k \) it appears that it should be expressed in terms of \( \Delta \text{BHP}_{jk} \), where \( \Delta \text{BHP}_{jk} = \text{BHP}_j - \text{BHP}_k \).

This, in turn, implies that we have calculated redundant terms in our previous examples. This redundancy was expressed by weights having the same magnitude and different sign. Note that in terms of the variance the behavior of the \( \text{BHP}_k \) and \( \Delta \text{BHP}_{jk} \) are the same for those cases where only one of the bottomhole pressures fluctuates.

Hence,

\[
\gamma_{jk} \Delta \text{BHP}_{jk} = \frac{A_{jk} k_{jk} \Delta P_{jk}}{\mu L_{jk}} \tag{3.12}
\]

the pressure terms cancel:

\[
\gamma_{jk} = \frac{A_{jk} k_{jk}}{\mu L_{jk}} \tag{3.13}
\]

or from Eq. (3.10):
\[ \gamma_{jk} = \frac{q_{jk}}{\Delta P_{jk}} \quad (3.14) \]

\( \gamma_{jk} \) is an explicit “interwell” productivity index term.

We can test expression (3.13) against the IBMLR+P results obtained in the previous section.

### 3.3.1. Testing the Physical Meaning of the Producer-Producer Weights

The dimensions of the linear reservoirs used here are summarized in Table 3-6. Permeability and viscosity properties are listed in Table 3-7.

<table>
<thead>
<tr>
<th>Table 3-6</th>
<th>Dimensions for the linear reservoirs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height: 60ft</td>
<td></td>
</tr>
<tr>
<td>Width: 80 ft</td>
<td></td>
</tr>
<tr>
<td>Area: 60ft x 80ft = 4800 ft²</td>
<td></td>
</tr>
<tr>
<td>Length: 30 x 80ft = 2400 ft</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 3-7</th>
<th>Permeability and viscosity properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Viscosity [cP]</td>
<td>Water</td>
</tr>
<tr>
<td>Endpoint relative permeability</td>
<td>0.9</td>
</tr>
<tr>
<td>Absolute permeability [md]</td>
<td>400</td>
</tr>
</tbody>
</table>

For field units a unit conversion factor is introduced into Darcy equation (3.10):

\[ q_{jk} = 1.127 \times 10^{-3} \frac{A_{jk} k_{jk} \Delta P_{jk}}{\mu L_{jk}} \quad (3.15) \]

and

\[ \gamma_{jk} = 1.127 \times 10^{-3} \frac{A_{jk} k_{jk}}{\mu L_{jk}} \quad (3.16) \]

where,

\([q] = \text{bbl/d}\)
\[ [A] = \text{ft}^2 \]
\[ [L] = \text{ft} \]
\[ [k] = \text{md} \]
\[ [\Delta P] = \text{psi} \]
\[ [\mu] = \text{cP} \]

Application of Eq. (3.16) to the linear homogeneous reservoir at 100% water saturation and 100% oil saturation results in:
\[
\gamma^w = 1.127 \times 10^{-3} \frac{4800 (0.225)(400)}{0.5(2400)} = 0.41
\]
\[
\gamma^o = 1.127 \times 10^{-3} \frac{4800 (0.9)(400)}{2(2400)} = 0.41
\]

which compare favorably with the producer-producer weights obtained from the regression model for the homogeneous cases summarized in Table 3-8.

### Table 3-8 Producer-producer weights for the linear homogeneous reservoir cases.

<table>
<thead>
<tr>
<th>Case 1211</th>
<th>Case 1211-1</th>
<th>Case 1212-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma_{11} )</td>
<td>-0.40</td>
<td>-0.40</td>
</tr>
<tr>
<td>( \gamma_{21} )</td>
<td>0.40</td>
<td>0.40</td>
</tr>
</tbody>
</table>

To test on the heterogeneous reservoir cases, the harmonic average permeability between the producers is first computed:

\[
\bar{k} = \frac{30}{8.5/400 + 21.5/920} = 672
\]

Again the producer-producer weights are calculated at 100% water saturation and 100% oil saturation according to Eq. (3.16):

\[
\gamma^w = 1.127 \times 10^{-3} \frac{4800 (0.225)(672)}{0.5(2400)} = 0.68
\]
\[ \gamma^0 = 1.127 \times 10^{-3} \times \frac{4800}{2} (0.9)(672) = 0.68 \]

These compare favorably with the producer-producer weights obtained from the regression model for the homogeneous cases summarized in Table 3-9.

**Table 3-9** Producer-producer weights for the linear heterogeneous reservoir cases.

<table>
<thead>
<tr>
<th>Case 1211-2</th>
<th>Case 1216-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma_{11} )</td>
<td>-0.67</td>
</tr>
<tr>
<td>( \gamma_{21} )</td>
<td>0.67</td>
</tr>
<tr>
<td>( \gamma_{21} )</td>
<td>-</td>
</tr>
<tr>
<td>( \gamma_{22} )</td>
<td>-</td>
</tr>
</tbody>
</table>

Thus, producer-producer weights obtained from regression reflect actual transmissibilities between producers.

### 3.4. NUMERICAL EXPERIMENTS CONTINUED

With these new insights into the producer-producer weights, we resume the numerical experiments.

#### 3.4.1. One Dimensional Numerical Experiments - Continued.

The last 1D experiment involves testing the revised model according to Eq. (3.10) and Eq. (3.12).

**a. Case 1216-2: Heterogeneous reservoir. More than one fluctuating BHP.**

Case 1216-2 is solved for in terms of the difference of BHP between producer P1 and P2 instead of BHP\(_1\) and BHP\(_2\).
Figure 3-10  Net bottomhole pressures difference for producers P1 and P2. Gaps in dataset correspond to periods where the BHPs were greater than the local reservoir pressure around the producers.

The producer-producer weights $\gamma$ are in agreement with those previously calculated for the linear heterogeneous reservoir. The signs indicate that the rate one producer gains the other loses. For this case, IBMLR+$\Delta P$ requires only two $\gamma$ related terms instead of four for the IBMLR+$P$ system (Table 3-5).

Table 3-10 Case 1216-2. Comparison of weights

<table>
<thead>
<tr>
<th>IDWK</th>
<th>IBMLR</th>
<th>IBMLR+$\Delta P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{11}$</td>
<td>0.50</td>
<td>$\beta_{11}$</td>
</tr>
<tr>
<td>$\beta_{21}$</td>
<td>0.50</td>
<td>$\beta_{21}$</td>
</tr>
<tr>
<td>$\mu_1$</td>
<td>-18.31</td>
<td>$\mu_1$</td>
</tr>
<tr>
<td>$\gamma_1\Delta p_{12}$</td>
<td>-0.65</td>
<td></td>
</tr>
<tr>
<td>$\gamma_2\Delta p_{12}$</td>
<td>0.65</td>
<td></td>
</tr>
<tr>
<td>$\mu_{p1}$</td>
<td>-50.27</td>
<td></td>
</tr>
</tbody>
</table>
3.4.2. Two Dimensional Numerical Experiments.

This set of experiments is aimed at testing the IBMLR+P system Eq. (3.7) in the 5x4 Synfield reservoirs used previously for testing IBMLR under constant BHP conditions. Starting from these cases, unless otherwise stated, the horizontal permeability is 400 md and the vertical permeability 40 md.

a. Case 1010: Homogeneous 5x4 Synfield

The input bottomhole pressure for producer P1 is shown in Figure 3-1. Unless otherwise stated, all other producers operate at constant BHP.

![Figure 3-1](image-url) Bottomhole pressure behavior used as input for the 5x4 Synfield cases.
Figure 3-12 Results of applying IBMLR (left) and IBMLR+P (right) when BHP fluctuates. The IBMLR+P weights are consistent with constant BHP IBMLR weights.

Figure 3-12 shows crossplots of IBMLR weights for constant BHP case versus IBMLR (left) and IBMLR+P (right) weights for the fluctuating BHP case. Recall that we have shown that the IBMLR results for constant BHP are equivalent to IDW estimates and single injector allocations (see sections 2.3 and 2.4). While applying conventional IBMLR analysis on the varying BHP case results in deviations from the physical estimates, IBMLR+P results in weights that agree with their physical estimates.

Producer-producer IBMLR+P $\gamma$ weights results are posted on Table 3-12. The sign of $\gamma_{11}$ is negative meaning that an increase in BHP$_1$ implies a decrease in P1 rate and vice versa. The magnitude of $\gamma_{11}$ is equal to the sum of $\gamma_{21}$, $\gamma_{31}$, $\gamma_{41}$. A unit change in producer P1 rate caused by a change in BHP$_1$ causes opposite changes in producers P2, P3, P4 rates according to Table 3-11.
Table 3-11 Case 1010. Normalized $\gamma$ weight ratios.

<table>
<thead>
<tr>
<th>$\gamma_{j1}$</th>
<th>$\frac{2.41}{6.61}$ ≡ 0.36</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_{31}$</td>
<td>$\frac{2.41}{6.61}$ ≡ 0.36</td>
</tr>
<tr>
<td>$\gamma_{41}$</td>
<td>$\frac{1.79}{6.61}$ ≡ 0.27</td>
</tr>
</tbody>
</table>

Table 3-12 Case 1010: Producer-producer IBMLR+P weights.

<table>
<thead>
<tr>
<th>$\gamma_{j1}$</th>
<th>P1</th>
<th>P2</th>
<th>P3</th>
<th>P4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_{11}$</td>
<td>-6.61</td>
<td>2.41</td>
<td>2.41</td>
<td>1.79</td>
</tr>
</tbody>
</table>

The $\gamma$ values for producers P2, P3 and P4 are in agreement with their relative position with respect to P1 (whose pressure fluctuates); with P2 and P3 receiving the same value being equidistant from P1, and P4, being the farthest away, receiving the lowest value. From expression Eq. (3.13), the weights should be inversely proportional to the distance. We can test this relationship between the $\gamma$ weights for this case:

$$\frac{\gamma_{21}}{\gamma_{41}} = \frac{L_{41}}{L_{21}}$$

Recall that interwell spacing is 800ft:

$$\frac{\gamma_{21}}{\gamma_{41}} = \frac{2.41}{1.79} ≡ 1.35 \quad \text{and} \quad \frac{L_{41}}{L_{21}} = \frac{2(800)}{\sqrt{800^2 + 800^2}} ≡ 1.41$$

The difference between both estimates is less than 5% with respect to the distance ratio.
b. **Case 1011: Homogeneous 5x4 Synfield –Shut-in Period**

The purpose of this case is to show that the IBMLR+P model is able to handle one of the limitations of the IBMLR model. A shut-in producer is a particular case of an extreme change in BHP where this one tends to the reservoir pressure (Figure 3-13).

![Figure 3-13 Case1011. P1 is shut-in for 180 days (left). BHP is equal to reservoir pressure over the same period of time (right).](image)

Application of IBMLR to this dataset would have resulted in unrealistically large and small weights with extreme deviations from their expected values as depicted in Figure 3-14 (left). Figure 3-14 (right) shows that the use of the IBMLR+P model results in weights that are in agreement again with their expected values.
Figure 3-14 Case 1011 weight comparison. IBMLR model results in non-physical weights (left). IBMLR+P estimates agree with constant BHP case.

Producer-producer $\gamma$ values obtained from IBMLR+P agree with the previous case (Table 3-13 and Table 3-12), as expected since the reservoir properties and well configuration have remained the same.

Table 3-13 Case 1011: Producer-producer IBMLR+P weights.

<table>
<thead>
<tr>
<th></th>
<th>P1</th>
<th>P2</th>
<th>P3</th>
<th>P4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_{ij}$</td>
<td>-6.61</td>
<td>2.41</td>
<td>2.41</td>
<td>1.78</td>
</tr>
</tbody>
</table>

c. Case 1020: 5x4 Synfield – More than one fluctuating BHP

The IBMLR+P model successfully handles cases where more than one well has fluctuating bottomhole pressures (Figure 3-15). Again, $\beta$ weights regressed match their physical estimates (and the weights corresponding to same case with constant bottomhole pressure). Producer-producer weights, representing the productivity index between producer pairs, are consistent with the previously presented homogeneous cases (Table 3-14).
**Figure 3-15** Bottomhole pressures used as input for the case 1020.

**Figure 3-16** Case 1020. Results of applying IBMLR (left) and IBMLR+P (right) when two BHPs fluctuate. IBMLR+P weights are consistent with constant BHP IBMLR weights.

**Table 3-14** Producer-producer $\gamma$ values obtained from IBMLR+P agree with the previous homogeneous cases.

<table>
<thead>
<tr>
<th></th>
<th>P1</th>
<th>P2</th>
<th>P3</th>
<th>P4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_1$</td>
<td>-6.61</td>
<td>2.40</td>
<td>2.41</td>
<td>1.79</td>
</tr>
<tr>
<td>$\gamma_2$</td>
<td>2.41</td>
<td>-6.60</td>
<td>1.78</td>
<td>2.41</td>
</tr>
</tbody>
</table>
d. Case 2110: Heterogeneous 5x4 Synfield

The IBMLR+P model is tested on the heterogeneous 5x4 Synfield (Figure 2-27) with producer P1 BHP fluctuating according to Figure 3-11.

Producer-producer weights $\gamma$ presented in Table 3-15 show the same behavior with respect to relative distance as in previous cases 1010 and 1011. $\gamma_{21}$ and $\gamma_{31}$ receive the same value while $\gamma_{41}$ receives the lowest value. This is also apparent in the weight ratios (Table 3-16), which show similar results as in the homogeneous case 1010 (Table 3-11). Average permeability of the heterogeneous field is lower than in the homogeneous field and this is reflected in the $\gamma$ weights, which have smaller values than in previous cases.

![Figure 3-17 Case 2110, Heterogeneous 5x4 Synfield. Results of applying IBMLR (left) and IBMLR+P (right) when BHPs fluctuate. IBMLR+P weights are consistent with constant BHP IBMLR weights.](image)

Table 3-15 Case 2110. Producer-producer $\gamma$ values obtained from IBMLR+P.

<table>
<thead>
<tr>
<th></th>
<th>P1</th>
<th>P2</th>
<th>P3</th>
<th>P4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_{1}$</td>
<td>-1.86</td>
<td>0.68</td>
<td>0.68</td>
<td>0.50</td>
</tr>
</tbody>
</table>
Recall that in section 3.1 an ordinary cokriging-like model using BHP (IBMLR+P) as the secondary variable had initially been proposed and later it was suggested that the difference in BHP (IBMLR+ΔP) would be a more appropriate choice in as much as this was consistent with the interpretation of the γ weights as productivity indexes. To further support this notion, the regression results for P1 and its estimate are presented. Figure 3-18 shows P1 data and its estimate as per the IBMLR+P model. The P1 estimate is clearly biased. The bias is explained by the fact that the average BHP for P1 (~200 psia) is lower than the constant operating BHP of the other producers (500 psia). Using the same weights listed in Table 3-15 but using $\gamma_{11} \times \Delta BHP$ (BHP1 – 500 psia) in the estimate instead of $\gamma_{11} \times BHP1$ results in an unbiased estimate for P1 (Figure 3-19).

Table 3-16 Case 2110. Normalized γ weight ratios.

<table>
<thead>
<tr>
<th>$\gamma_{21}$</th>
<th>$\gamma_{11}$</th>
<th>$\gamma_{31}$</th>
<th>$\gamma_{11}$</th>
<th>$\gamma_{21}$</th>
<th>$\gamma_{11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.68</td>
<td>0.37</td>
<td>0.68</td>
<td>0.37</td>
<td>0.50</td>
<td>0.27</td>
</tr>
<tr>
<td>1.86</td>
<td>1.86</td>
<td>1.86</td>
<td>1.86</td>
<td>1.86</td>
<td>1.86</td>
</tr>
</tbody>
</table>
Figure 3-18 Case 2110. P1 historic data and corresponding estimate. Estimate is biased.

Figure 3-19 Case 2110. P1 historic data and corresponding estimate. Bias is corrected by accounting for the difference in pressure between producers.

3.5. CONCLUSIONS

A generic model that incorporates bottomhole pressure information has been developed. This model overcomes the constant bottomhole pressure limitation of the MLR, ABMLR and IBMLR models. Not only does the model successfully estimate
the rates, but it gives a set of weights that are consistent with the IBMLR weights and their physical interpretation.

In addition, the new producer-producer weights obtained also contain a physical meaning. This is in terms of absolute productivity indexes, whereas IBMLR weights are relative magnitudes.

Finally, the volume between producers is explicitly probed resulting in more areal coverage of the reservoir properties.

### 3.6. RECOMMENDATIONS FOR FUTURE WORK

Reformulate the IBMLR+P system Eq. (3.7) in terms of the difference in pressure between producers. In particular, redundant weights (e.g. $\gamma_{12} = -\gamma_{21}$) that associated to the same well pair could be eliminated. Also, auto-referencing $g$ weights (e.g. $\gamma_{11}$) could be replaced by the sum of the $\gamma$ weights associated with said producer.

Also, reformulate the IBMLR system in term of the injection pressure rather than injection rates. This would probably result in injector-producer weights comparable to productivity indexes or transmissibility terms.
CHAPTER 4: HIGH-ORDER ALLOCATION MODELS

So far, I have described how the MLR, ABMLR, IBMLR, and IBMLR+P\textsuperscript{4} models estimate production rates making use of injection rates (and bottomhole pressures in the case of IBMLR+P). Regression coefficients, $\beta$, describe how each injection rate is related to the estimated production rate and can be related to the reservoir characteristics such as permeability. However, these models do not describe how the water is physically transported from injectors to producers. Such a model could be combined with saturation versus water-cut models to enable oil phase regression (chapter 5) and subsequent field (oil phase) optimization.

The following are attempts to generate such a model based on the idea of that the weights should physically represent the fraction of water allocated from neighboring injectors to producers. Hence, the models, by construction, are based solely on explicit relationships between neighboring injectors and producers. However, allocation weights will carry information about injector interactions. This, in turn, will give rise to high-order moments in the system of equations.

\textsuperscript{4} MLR: Multivariate linear regression.
ABMLR: Average balanced multivariate linear regression.
IBMLR: Instantaneously balanced multivariate linear regression.
IBMLR+P: Instantaneously balanced multivariate linear regression including bottomhole pressure.
4.1. ALLOCATION MODEL

The allocation model starts very simple. Its basic premise is that the producing rate of any one producer is exclusively a linear combination of the rates of its immediate neighboring injectors.

\[ P_j = \sum_{i_{\text{Adjacent}}} B_{ji} I_i \]  

(4.1)

where \( I_i \) represents the rate of each of the neighboring injectors of the pattern corresponding to \( P_j \), \( B_{ji} \) is the allocation factor of an injector \( i \) to its neighboring producer \( j \). \( i_{\text{Adjacent}} \) is a list of injectors adjacent to producer \( j \).

Although Eq. (4.1) resembles a particular case of MLR, in the allocation models we let the \( B_{ji} \) weights be a linear combination of the injector \( i \) rate, the non-neighboring injector rates, and a constant (Eq. (4.2)).

\[ B_{ji} = \beta_{i0} + \beta_{ii} I_i + \sum_m \beta_{im} I_m \]  

(4.2)

where

\( m \) is the list of injectors non-adjacent to producer \( j \).
and the \( \beta \) are regression coefficients that describe the interference between injector \( i \) and non adjacent injectors \( m \).

The implications for the model are the following:

- The number of regression weights, for the 5x4 Synfield increase from five to nine per producer or in general \((i_{\text{Adjacent}} \times (m+2))\). Although in practice \( m \) can be very large, it is suggested to use non-adjacent injectors no farther than the second row of injectors around a producer. This restriction on \( m \) would prevent unrealistic relationships between far away injectors as well as preventing a large number of
weights (for a large field non-adjacent injectors could be in the order of hundreds) that would, in turn, lower the over-determination of the problem or might even render it undetermined.

-Regression variables will involve a combination of the product of rates from neighboring and non-neighboring injectors. This, in turn, will give rise to high-order moments and interactions between neighboring and non-neighboring injectors.

It is the intention, however, that by keeping the general representation of a linear combination of neighboring injector rates through a simple $B_{ji}$ weight the interpretation of the regression coefficients will remain physically tractable and can still be thought of as an allocation model. Instead of having to interpret many $\beta$s, static coefficients associated with complex multiple injector interactions, we can instead interpret the behavior of $B$ terms of an allocation concept. Note that $B$ is dynamic i.e. it changes with time.

4.1.1. 5×4 Synfield Model Derivation

Figure 4-1 presents a schematic of the flow in the 5×4 Synfield and the corresponding $B_{ji}$ weights.

For $P_1$:
\[ \hat{P}_1 = B_{11} I_1 + B_{12} I_2 + B_{13} I_3 \] (4.3)

Similarly for the other producers:
\[ \hat{P}_2 = B_{21} I_1 + B_{22} I_2 + B_{24} I_4 \] (4.4)
\[ \hat{P}_3 = B_{32} I_2 + B_{33} I_3 + B_{35} I_5 \] (4.5)
\[ \hat{P}_4 = B_{43} I_3 + B_{44} I_4 + B_{45} I_5 \] (4.6)
Figure 4-1 5x4 Synfield well layout. Arrows indicate flow and allocation between injectors and producers and corresponding $B_{ji}$ weight.

Figure 4-2 shows the relationship between $B_{ji}$ weights and non-neighboring injectors. and the $B_{ji}$ terms are thus:

$$B_{11} = \beta_{11} I_1 + \beta_{14} I_4 + \beta_{15} I_5 + \beta_{10}$$  \hspace{1cm} (4.7)

$$B_{12} = \beta_{22} I_2 + \beta_{24} I_4 + \beta_{25} I_5 + \beta_{20}$$  \hspace{1cm} (4.8)

$$B_{13} = \beta_{33} I_3 + \beta_{34} I_4 + \beta_{35} I_5 + \beta_{30}$$  \hspace{1cm} (4.9)
Figure 4-2 Allocation model schematic. Schematic showing how non-neighboring injectors interact with the producer through the $B_{ij}$ weight associated with the neighboring injector.

Just as with the previous models, to obtain the regression coefficients we can either minimize the expectation of the squared residuals or minimize the variance of the residuals. Both approaches are presented next.

4.1.2. Minimizing the Expectation of the Squared Residuals

We start with expressing the expectation of the squared residual in terms of the $\beta_{ji}$ weights for producer $P_1$:

$$E\left[ (P_1 - \hat{P}_1)^2 \right] = E\left[ P_1^2 - 2P_1(B_{11}I_1 + B_{12}I_2 + B_{13}I_3) + (B_{11}I_1 + B_{12}I_2 + B_{13}I_3)^2 \right]$$

(4.10)

Substituting equations (4.7) through (4.9) in Eq. (4.10):
To minimize this function we again set the partial derivatives with respect to the \( \beta \) coefficients to zero. This is presented in detail for two of the weights after which a general answer is presented.

\[
\frac{\partial E[\hat{\beta}]}{\partial \beta_{10}} = E \left( \begin{array}{l}
2I_1^2\beta_{11} + 2I_1^2I_4\beta_{14} + 2I_1^2I_5\beta_{15} + 2I_1^2\beta_{10} + 2I_1I_3^2\beta_{22} \\
+2I_1I_2I_4\beta_{24} + 2I_1I_2I_5\beta_{25} + 2I_1I_2\beta_{20} + 2I_1I_3^2\beta_{33} \\
+2I_1I_4I_3\beta_{34} + 2I_1I_3I_5\beta_{35} + 2I_1I_3\beta_{30} - 2I_1P_1
\end{array} \right) = 0 \quad (4.12)
\]

\[
\frac{\partial E[\hat{\beta}]}{\partial \beta_{34}} = E \left( \begin{array}{l}
2I_3^2I_3^2\beta_{11} + 2I_3^2I_4^2I_3\beta_{14} + 2I_3I_4I_5\beta_{15} + 2I_3I_4I_3\beta_{10} \\
+2I_3I_4I_2^2\beta_{22} + 2I_3I_4I_2I_5\beta_{24} + 2I_3I_4I_2I_5\beta_{25} + 2I_3I_4I_2I_3\beta_{20} \\
+2I_3I_4I_3^2\beta_{33} + 2I_3^2I_4^2\beta_{34} + 2I_3^2I_4I_5\beta_{35} + 2I_3^2I_5\beta_{30} + 2I_3I_4P_1
\end{array} \right) = 0 \quad (4.13)
\]

Similar equations can be derived for the other weights.

The resulting (12\times12) system of equations in matrix form for producer P1 is:
The system Eq. (4.14) combines 2nd through 4th order moments. Note that the ordering of indexes of the elements matrix is similar to the ordering of indexes of the vector of weights. The indexes of the injectors involved in the expectation operator are those corresponding to the indexes of the weight in that same row and those of the weight located in a row equal to the column number of the element. For example:

To determine the element corresponding to the second row and seventh column of the matrix we refer to:

the weight located in the second row: $\beta_{11} \Rightarrow I_1 I_1 = I_1^2$

the weight located in the seventh row: $\beta_{24} \Rightarrow I_2 I_4$

Hence the element is $E(I_1^2 I_2 I_4)$
This observation allows a generalization of the system of equations for any configuration of adjacent and non-adjacent injector-producer pairs chosen.

4.1.3. Model Generalization

The model can be generalized by re-indexing the rows and columns of the matrix representing the corresponding system of equations. The four-digit index of the elements corresponding to the left matrix is formed by combining the two-digit row index with the two-digit column index. The row and column indexes, in turn, are defined by the “ik” indexes of the vector of $\beta_{ik}$ weights. This is illustrated by the system of equations (4.15). The value of elements of the matrix can be calculated by applying operators Eq. (4.16) and Eq. (4.17). When any one sub-index is zero the particular injection vector is dropped and the operator results in a lower order moment. Recall that to determine the k index the rules given by Eq. (4.2) should be followed and that the index $i$ is determined by the injector adjacent to $B_{ji}$.

Likewise, the producer-injector moment matrix is form by combining producer j and the injectors determined by the $\beta$ weight located in the same row (Eq. (4.17)).

\[
\begin{pmatrix}
\beta_{10} & \beta_{11} & \beta_{14} & \beta_{15} & \cdots & \beta_{ik}
\end{pmatrix}
\begin{pmatrix}
a_{1010} & a_{1011} & a_{1014} & a_{1015} & \cdots & a_{10ik}
\end{pmatrix}
\begin{pmatrix}
\beta_{10} \\
\beta_{11} \\
\beta_{14} \\
\beta_{15} \\
\vdots \\
\beta_{ik}
\end{pmatrix}
= 
\begin{pmatrix}
b_{10} \\
b_{11} \\
b_{14} \\
b_{15} \\
\vdots \\
b_{ik}
\end{pmatrix}
\] (4.15)
\[ a_{wxyz} = E(I_w I_x I_y I_z) \]  
\[ b_{ik} = E(P_j I_i I_k) \]  

4.1.4. Minimizing the Variance of the Residuals

Alternatively, instead of minimizing the expectation of the squared error, the variance of the error (Eq. (4.18)) can be minimized.

\[ \text{Var}(P_j - \hat{P}_j) = E\left[ (P_j - \hat{P}_j)^2 \right] - E^2\left[ P_j - \hat{P}_j \right] \]  
\[ (4.18) \]

This minimization results in a similar matrix to Eq. (4.15) but the resulting elements of the matrix are centered moments defined according to Eq. (4.19).

\[ a_{jklm} = E(I_j I_k I_l I_m) - E(I_j) E(I_k) E(I_l) E(I_m) \]  
\[ (4.19) \]

\[ b_{ij} = E(P_i I_i I_j) - E(P_i) E(I_i) E(I_j) \]  
\[ (4.20) \]

4.1.5. Inference of High-Order Moments

Higher order moments Eqs. (4.16), (4.17), (4.19) and (4.20) can be calculated explicitly from the injection-production rates dataset. A new variable composed of the product of the corresponding variables associated to the moment is calculated for every historic time level. Next the expectation of this new variable is approximated by a time average.

\[ E\left( I_w I_x I_y I_z \right) = \frac{1}{M} \sum_{m=1}^{M} (I_w I_x I_y I_z)^{(m)} \]  
\[ (4.21) \]

where \( m \) indicates the time level and \( M \) is the total number of historic time levels.

Other high-order moments can be calculated similarly.
4.1.6. Additional Injection Balance Constraints

For this model we can also require a balance for each of the injectors as for IBMLR that enables the interpretation of the “B” weights as allocation-like factors. We require the sum of the allocation factors for a given injector to be equal to one. Since the weights $B_{ji}$ are a function of the injection rates which can be considered random variables, the time variation of $B_{ji}$ is stochastic. Hence, the balance condition will be expressed in terms of expectations which is simply a time average.

$$\sum_j e_i E(B_{ji}) = 1 \quad \text{for all } i \quad (4.22)$$

where

$B_{ji}$ are defined according to neighboring and non-neighboring relationships between the injectors and producers Eq. (4.7).

e$_i$ is an injection efficiency factor to account for losses or leaks in the system.

For the 5x4 Synfield (Figure 4-3) where $e_i=1$, Eq. (4.22) results in:

$$E(B_{11}) + E(B_{21}) = 1$$
$$E(B_{12}) + E(B_{32}) = 1$$
$$E(B_{13}) + E(B_{23}) + E(B_{33}) + E(B_{43}) = 1$$
$$E(B_{24}) + E(B_{44}) = 1$$
$$E(B_{35}) + E(B_{45}) = 1 \quad (4.23)$$

These balance conditions, in turn, translate into a set of auxiliary equations in terms of the $\beta$ weights. These auxiliary equations are added into the system of equations with Lagrange multipliers, which couple otherwise independent systems of equations, one for each producer.

For the 5x4 Synfield (Figure 4-3) the injection balance equations are (the superscript refers to the producer number):
\[ E(B_{11}) + E(B_{21}) = 1 \]
\[ \beta_{10}^1 + \beta_{11}^1 E(I_1) + \beta_{14}^1 E(I_4) + \beta_{15}^1 E(I_5) + \beta_{10}^2 + \beta_{11}^2 E(I_1) + \beta_{12}^2 E(I_2) + \beta_{13}^2 E(I_3) + \beta_{14}^2 E(I_4) + \beta_{15}^2 E(I_5) = 1 \] (4.24)

\[ E(B_{12}) + E(B_{32}) = 1 \]
\[ \beta_{20}^1 + \beta_{22}^1 E(I_2) + \beta_{24}^1 E(I_4) + \beta_{25}^1 E(I_5) + \beta_{20}^3 + \beta_{21}^3 E(I_1) + \beta_{22}^3 E(I_2) + \beta_{23}^3 E(I_3) + \beta_{24}^3 E(I_4) = 1 \] (4.25)

\[ E(B_{13}) + E(B_{23}) + E(B_{33}) + E(B_{43}) = 1 \]
\[ \beta_{30}^1 + \beta_{33}^1 E(I_3) + \beta_{34}^1 E(I_4) + \beta_{35}^1 E(I_5) + \beta_{30}^2 + \beta_{32}^2 E(I_2) + \beta_{33}^2 E(I_3) + \beta_{35}^2 E(I_5) + \beta_{30}^3 + \beta_{31}^3 E(I_1) + \beta_{32}^3 E(I_2) + \beta_{33}^3 E(I_3) + \beta_{34}^3 E(I_4) + \beta_{35}^3 E(I_5) = 1 \] (4.26)

\[ E(B_{24}) + E(B_{44}) = 1 \]
\[ \beta_{40}^2 + \beta_{42}^2 E(I_2) + \beta_{44}^2 E(I_4) + \beta_{45}^2 E(I_5) + \beta_{40}^4 + \beta_{41}^4 E(I_1) + \beta_{42}^4 E(I_2) + \beta_{43}^4 E(I_3) + \beta_{44}^4 E(I_4) = 1 \] (4.27)
E(B_{35}) + E(B_{45}) = 1
\beta_{50}^3 + \beta_{51}^3 E(I_1) + \beta_{54}^3 E(I_4) + \beta_{55}^3 E(I_5) + \beta_{50}^4 + \beta_{51}^4 E(I_1) + \beta_{52}^4 E(I_2) + \beta_{55}^4 E(I_5) = 1 \quad (4.28)

Figure 4-3 Allocation for each injector (green or dark color lines) should in average sum to one.

4.1.7. 5x4 Homogeneous Synfield Results

a. Rate Regression Results

Four variations of the allocation model can be applied by opting between centered or non-centered moments and between applying the balance constraint or not. The rate regression results for the homogeneous 5x4 Synfield are presented in Figure 4-4 and Figure 4-5.
Figure 4-4 Homogeneous 5x4 Synfield. Non-centered moments. The actual rate values on the x-axis are plotted against the regression results on the y-axis for the four producers. Left: not constrained; Right: balance constraint applied.

Figure 4-5 Homogeneous 5x4 Synfield. Centered moments. The actual rate values on the x-axis are plotted against the regression results on the y-axis for the four producers. Left: not constrained; Right: balance constraint applied.

The coefficients of determination are large for both centered and non-centered moments versions of the allocation regression model (Table 4-1, Figure 4-4, Figure 4-5).

Table 4-1 Coefficient of determination $R^2$ comparison for centered and non-centered allocation regression models applied to 5x4 Homogeneous Synfield case 15

<table>
<thead>
<tr>
<th>Centered and Balance Constraint</th>
<th>Non Centered and Balance Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>P1</td>
</tr>
<tr>
<td>P2</td>
<td>P2</td>
</tr>
<tr>
<td>P3</td>
<td>P3</td>
</tr>
<tr>
<td>P4</td>
<td>P4</td>
</tr>
<tr>
<td>0.97</td>
<td>0.97</td>
</tr>
<tr>
<td>0.98</td>
<td>0.98</td>
</tr>
<tr>
<td>0.98</td>
<td>0.98</td>
</tr>
<tr>
<td>0.99</td>
<td>0.99</td>
</tr>
</tbody>
</table>
b. Tracer vs. Weights Results

The focus of this section is the assessment of the weights as allocation factors. For validation of the model we resort to tracer injection. Non-reactive aqueous phase tracers were injected continuously into each injector in the simulated $5 \times 4$ Synfield, a different tracer in each injector. The concentration of the tracers injected was equal to one. This allows tracking of flow path followed by the injection water. After breakthrough and when the water oil ratio is high, the ratio of total mass of tracer produced at each producer to the total mass injected is approximately the fraction of water flowing between each injector-producers pair. These values are compared to the time average weights $B_{ji}$.

Table 4-2 Time-averaged allocation weights versus tracer fraction. Homogeneous 5x4 Synfield. ($R^2$ between tracer fraction and average weights).

<table>
<thead>
<tr>
<th></th>
<th>Non-centered Moments</th>
<th>Centered Moments</th>
<th>Tracer Fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[B_{11}]$</td>
<td>-0.24</td>
<td>0.49</td>
<td>0.68</td>
</tr>
<tr>
<td>$E[B_{12}]$</td>
<td>0.97</td>
<td>0.49</td>
<td>0.47</td>
</tr>
<tr>
<td>$E[B_{13}]$</td>
<td>0.53</td>
<td>0.25</td>
<td>0.05</td>
</tr>
<tr>
<td>$E[B_{21}]$</td>
<td>0.32</td>
<td>0.51</td>
<td>0.70</td>
</tr>
<tr>
<td>$E[B_{22}]$</td>
<td>0.54</td>
<td>0.23</td>
<td>0.31</td>
</tr>
<tr>
<td>$E[B_{23}]$</td>
<td>0.42</td>
<td>0.50</td>
<td>0.23</td>
</tr>
<tr>
<td>$E[B_{24}]$</td>
<td>0.49</td>
<td>0.51</td>
<td>1.45</td>
</tr>
<tr>
<td>$E[B_{31}]$</td>
<td>0.31</td>
<td>0.22</td>
<td>-0.56</td>
</tr>
<tr>
<td>$E[B_{32}]$</td>
<td>0.46</td>
<td>0.52</td>
<td>0.22</td>
</tr>
<tr>
<td>$E[B_{33}]$</td>
<td>0.49</td>
<td>0.30</td>
<td>0.75</td>
</tr>
<tr>
<td>$E[B_{34}]$</td>
<td>0.37</td>
<td>0.50</td>
<td>-0.16</td>
</tr>
<tr>
<td>$E[B_{35}]$</td>
<td>0.44</td>
<td>0.48</td>
<td>0.73</td>
</tr>
<tr>
<td>$R^2$</td>
<td>-0.129</td>
<td>0.986</td>
<td>0.412</td>
</tr>
</tbody>
</table>

Although the four variations of the allocation model yielded good regression results, from a weight behavior point of view, the constrained model using non-centered moments is the one that more closely reproduces the tracer fraction produced.
Moreover, the balance constraint seem to be necessary to keep the average of the weights within physical bounds (0 to 1).

Figure 4-6 Tracer fractions versus time averaged allocation weights. Non-centered moments and balance constraint. Homogeneous 5x4 Synfield.

c. Balance Constrained Weight Behavior

Figure 4-6 shows time-averaged values for the allocation weights E(B_{ji}). In this section the time behavior of the allocation weight is presented for the conditions that best reproduce the tracer fractions: a balance constraint and non-centered moments.

The time behavior of the pseudo-allocation weight B_{ji} is presented in Figure 4-7. Although the true time behavior of the allocation weights is not known, we can still make some observations.
Figure 4-7 Time behavior of the allocation weights. Balance constraint and non-centered moments. 5x4 homogeneous Synfield.

1. The allocation weights $B_{ij}$ fluctuate with the injection rates as anticipated by the model (4.2). Physically this implies that allocation depends on the combination of injection rates. Allocations shift around slightly as the pressure field varies with injection rates. From a regression point of view, allowing $B_{ji}$ to fluctuate with time results in a better match to the estimated production rate.
2. The oscillations are around constant mean values. As expected since they are a linear combination of the input injection signal, they also inherit the character of the injection profiles (Figure 4-8).

3. The more producers an injector feeds, the lower the mean. Moreover this mean seems to be close to $1/(\text{Number of neighboring producers})$ e.g. for injector I3 (all weights $B_{ji}$ with $i=3$ in Figure 4-7) it is $\frac{1}{4} = 0.25$; for all other injectors it is $\frac{1}{2} = 0.5$. These values are expected for a homogeneous reservoir where all injectors share a common injection pressure and all producers share a common flowing bottomhole pressure. More generally for a heterogeneous reservoir we would expect the allocations of such a balanced pressurized reservoir to be equal to the IBMLR weight ratios $R_{\beta_{ji}}$ defined in section 2.1.1.b:

$$R_{\beta_{ji}} = \frac{\beta_{ji}}{\sum_{j \text{Pattern}} \beta_{ji}}$$  \hspace{1cm} (2.11)
The weights $B_{ji}$ are cross-plotted in three figures (for reasons of space); Figure 4-9 through Figure 4-11. Some strong correlation and anti-correlation is apparent in these figures.
**Figure 4-9** Allocation weights cross-plots (continued). Balance constraint and non-centered moments. 5x4 homogeneous Synfield.
Weights corresponding to producers P1 (B₁₁) show relatively large correlation with the weights corresponding to P2 (B₂₁). These weights (B₁₁, B₂₁), in turn, show weak to no correlation with weights corresponding to producers P3 (B₃₁) and P4 (B₄₁). Likewise there is some correlation among the weights corresponding to producers P3
Weights corresponding to the same producer are always correlated. In particular there is always a strong negative correlation between the center injector I3 and at least one of the other two injectors directly associated with that producer (B11-B13, B21-B23, B43-B45).

<table>
<thead>
<tr>
<th>B33</th>
<th>B35</th>
<th>B43</th>
<th>B44</th>
<th>B45</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Graph" /></td>
<td><img src="image2.png" alt="Graph" /></td>
<td><img src="image3.png" alt="Graph" /></td>
<td><img src="image4.png" alt="Graph" /></td>
<td><img src="image5.png" alt="Graph" /></td>
</tr>
</tbody>
</table>

Figure 4-11 Pseudo-allocation weights crossplots (continued). Regression model 4b, balance constraint and non-centered moments. 5x4 homogeneous Synfield.

Although this interplay between weights is a desired behavior sought by this kind of model, physical validation is still lacking and should be a matter of further research. Streamline simulation models are again suggested for this purpose.

4.2. SIMPLIFIED MODEL

If all $\beta$ weights in Eq. (4.2), except the $\beta_{ib}$, are set to zero, injector-injector interactions disappear and the system reverts to a particular case of MLR where the rate produced by any one producer is exclusively a linear combination of the rates of
its immediate neighboring injectors. Thus, performance as a production rate estimator is expected to be poor. However, when the balance constraint is applied and non-centered moments used, the model still seems useful in terms of capturing an average allocation as illustrated in the following case.

### 4.2.1. 5x4 Homogeneous Synfield Results

#### a. Rate Regression Results

Rate regression results for the 5x4 homogeneous Synfield are presented in Figure 4-12. Except for the unconstrained centered system, the simplified models tend to under-predict small rates and over-predict large rates. However, this unconstrained model using centered moments is the worst performer in terms of tracer allocation (Table 4-4).

**Figure 4-12** 5x4 Synfield. Non-centered moments results. The actual rates on the x-axis are plotted against the regression results on the y-axis for the four producers. Left: not constrained. Right: constrained.
Table 4-3 $R^2$ rate regression results for non-centered moments. Similar results are obtained with centered moments.

<table>
<thead>
<tr>
<th>Balance Constraint</th>
<th>No Bal. Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1 0.72</td>
<td>0.70</td>
</tr>
<tr>
<td>P2 0.73</td>
<td>0.73</td>
</tr>
<tr>
<td>P3 0.86</td>
<td>0.87</td>
</tr>
<tr>
<td>P4 0.85</td>
<td>0.85</td>
</tr>
</tbody>
</table>

Figure 4-13 5x4 homogeneous Synfield. Centered moments results. The actual rates on the x-axis are plotted against the regression results on the y-axis for the four producers. Left: not constrained. Right: constrained

b. Tracer vs. Weights Results

The ratios of cumulative mass of tracers produced to that of cumulative tracers mass injected at each injector at the end of the simulation runs are listed along with the weights obtained from the regressions. Results from the injector balance-constrained cases compare favorably with the fractions of tracers produced (Table 4-4 and Figure 4-14). In particular, the non-centered moments and balanced results compare favorably to those corresponding to the full allocation model (Table 4-2, second column).

However, as seen before, the quality of the regression results ($R^2$) using these constrained models is only fair (Table 4-3). The models are unable to capture the
complexity of the signal at the producers from just the signal from the three immediate injectors.

Hence, the advantage of the full allocation regression model is that it lets the $B_{ji}$ weights vary instead of being constant. From a physical point of view this variation is expected to happen because as injection rates change so does the pressure field, forcing the streamline distributions to adjust accordingly.

Table 4-4 Simplified model. Regressed weights versus tracer fraction. Homogeneous 5x4 Synfield. ($R^2$ is between tracer fraction and weights)

<table>
<thead>
<tr>
<th></th>
<th>Non-centered Moments</th>
<th></th>
<th>Centered Moments</th>
<th></th>
<th>Tracer Fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_{11}$</td>
<td>0.64</td>
<td>0.47</td>
<td>0.34</td>
<td>0.41</td>
<td>0.49</td>
</tr>
<tr>
<td>$B_{12}$</td>
<td>0.33</td>
<td>0.49</td>
<td>0.28</td>
<td>0.48</td>
<td>0.49</td>
</tr>
<tr>
<td>$B_{13}$</td>
<td>0.26</td>
<td>0.27</td>
<td>0.20</td>
<td>0.26</td>
<td>0.23</td>
</tr>
<tr>
<td>$B_{21}$</td>
<td>0.68</td>
<td>0.53</td>
<td>0.41</td>
<td>0.59</td>
<td>0.51</td>
</tr>
<tr>
<td>$B_{23}$</td>
<td>0.23</td>
<td>0.22</td>
<td>0.19</td>
<td>0.23</td>
<td>0.25</td>
</tr>
<tr>
<td>$B_{24}$</td>
<td>0.32</td>
<td>0.49</td>
<td>0.28</td>
<td>0.50</td>
<td>0.49</td>
</tr>
<tr>
<td>$B_{32}$</td>
<td>0.44</td>
<td>0.51</td>
<td>0.31</td>
<td>0.52</td>
<td>0.51</td>
</tr>
<tr>
<td>$B_{33}$</td>
<td>0.33</td>
<td>0.26</td>
<td>0.21</td>
<td>0.27</td>
<td>0.25</td>
</tr>
<tr>
<td>$B_{35}$</td>
<td>0.49</td>
<td>0.48</td>
<td>0.34</td>
<td>0.49</td>
<td>0.49</td>
</tr>
<tr>
<td>$B_{43}$</td>
<td>0.35</td>
<td>0.25</td>
<td>0.20</td>
<td>0.24</td>
<td>0.27</td>
</tr>
<tr>
<td>$B_{44}$</td>
<td>0.39</td>
<td>0.51</td>
<td>0.27</td>
<td>0.50</td>
<td>0.51</td>
</tr>
<tr>
<td>$B_{45}$</td>
<td>0.54</td>
<td>0.52</td>
<td>0.35</td>
<td>0.51</td>
<td>0.51</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.657</td>
<td>0.986</td>
<td>0.854</td>
<td>0.951</td>
<td></td>
</tr>
</tbody>
</table>
4.3. HIGH ORDER ALLOCATION MODEL: EXTENDED TESTING

4.3.1. Faulted 5x4 Synfield

Regression results for the full allocation model are shown on Figure 4-15 for the faulted 5x4 Synfield presented in section 2.3.3.c, Figure 2-18. Except for outliers corresponding to the first few points of the historic data, the match is very good. Note however, that since the adjacent wells relationships were constructed assuming absence of faults, the model erroneously lacks allocation from I5 (Figure 2-18). This is physically incorrect and results from ignoring the presence of the fault at the time of constructing the model. Although regression results are satisfactory, this example
(and also case 500 in section 2.3.3.d) illustrates that these (and any) regression models are best used within a geological and physical framework.

![MinErr - Constrained](image)

**Figure 4-15** Model 4b regression results for the faulted 5x4 Synfield.

Recall that while IBMLR and IBMLR+P weights are solely a function of average transmissibilities between well pairs and, hence, independent of the dynamic state of the reservoir, allocation weights (tracer fractions) depend on the dynamic state. Almost no water injected in I3 is produced in P2 (Figure 4-16, right). Similarly, almost no tracer injected in I4 is produced in P2. However, since the reservoir is homogeneous, the transmissibilities between these pairs should be the same as can be seen on the inverse distance weight map (Figure 4-17). The reason for the apparent lack of communication between these wells is that all of the water injected in I1 is produced in P2, which diverts most of the water injected in I3 and in I4 towards P4.

Except maybe for the P2-I3 weight, model 4b (balance constrained and using non-centered moments) successfully captures the dynamics of the field (Figure 4-16 left vs. right).
Figure 4-16 Pseudo-allocation weights (left) and tracer fractions (right) for faulted 5x4 Synfield. The segments originating at the injectors are proportional to the weights and tracer fraction correspondingly.

Figure 4-17 Inverse distance weights for faulted 5x4 Synfield.
4.3.2. **Zig-Zag 25x16 Synfield**

A high permeability channel zigzags across the 25x16 Synfield in this case (Figure 4-18). Non-centered moments with the balance constraint were used on the allocation model. The allocation weights correlate with tracer fraction, however, the results are not good enough to be used in a quantitative manner (Figure 4-19, left). On the other hand, the tracer fraction results might be a poor indicator of injection allocation in this complex permeability field because breakthrough times and water-cut could differ significantly from one injector-producer pair to the other. The IBMLR weight ratios have been presented for comparison (Figure 4-19, right). These also show differences with respect to tracer fractions.

![Figure 4-18](image)

*Figure 4-18* Case 515: ZigZag pattern permeability field. Permeability is 40 md in the yellow (light) coded cells and 120 md in the red (dark) coded cells.
4.4. APPLICATION TO CHIHUIDO DE LA SIERRA NEGRA FIELD

This section shows the application of the high order allocation model to the Chihuido de la Sierra Negra (ChSN) field. Multivariate linear regression results for this same dataset are available in Albertoni 2002.

4.4.1. Field Description

From Albertoni 2002 we have the following brief field description:

“The Chihuido de la Sierra Negra (ChSN) Field is located in Western Argentina….”

“The ChSN field is undergoing a waterflood on a five-spot pattern. Fairly continuous eolian and river-channel sandstones constitute the reservoir, which has five main productive layers. The average depth is 1000 meters, the average net thickness is 20 meters, porosity ranges from 0.15 to 0.25, and the average permeability is 40 md. Oil and water compressibilities are moderate and the water-oil mobility ratio is approximately equal to one. The injectors have selective injection systems, where the injection rate per layer (or group of layers) is
controlled by a set of downhole mandrels and valves. The rate distribution in the injectors suffered changes during the waterflood. Most of the producers have undergone changes in the artificial lift system, during the waterflood, from rod-pump units to electro-submersible pumps.”

Of the more 1000 wells in the fields, the available dataset consists of 25 injectors and 16 producers (Figure 4-21). One consideration when applying the allocation model is that a change from rod-pump to electro-submersible pump brings about changes in BHP that the model does not currently contemplate. On the other hand, this information is not known as of this time, but can be tentatively identified by abrupt temporal changes in the mean producing rates.

**Figure 4-20** Injection rates profiles for ChSN field.
A second consideration is that the ChSN boundaries are open. To achieve balance an approximation is used; the rates on the lateral boundary injectors were halved and the rates of the corner injectors was divided by four. The second issue with open boundaries is that interactions with injectors outside this area are lost. The behavior of the allocation weights for boundary wells is expected to be more affected by this fact.

Finally, the injection rate profiles used in the ChSN field application example are shown in Figure 4-20.

![Figure 4-20](image)

**Figure 4-21** Chihuido de la Sierra Negra Field. Base map.

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4.4.2. Weight Results and Interpretation

The results are presented in Figure 4-22. Vectors originating at the injector and pointing towards the producer represent the allocation from that injector to the producer. The length of the vectors is proportional to the time average value of the weights. Since the balance constraint was applied, the sum of the allocation weights $B$ for a given injector is equal to one.

The representation of the weights is overlain on a structural map. One of the main features of the structural map is the presence of three faults. The trace of fault A runs south of C14, across C15 and vanishes north of C16. Fault B runs south of F14 and its continuation is suggested by a thin line in the NW-SE direction with a slight counter-clockwise rotation. Finally the tip of fault C shows NW of E14.

The weights shown in Figure 4-22 are strongly asymmetric. The symmetric sweep that is expected from the five-spot pattern does not seem to be present. This can be because of a combination of geologic features and differences or imbalances in the operation of the wells. Some of the weights, in blue (dark color), are negative. Since this is physically impossible, those connections with average negative values were successively omitted from the model.
Figure 4-22 Chihuido de la Sierra Negra field. Allocation results. The length of the vectors is proportional to the time average value of the weights.

The results of this procedure are shown in Figure 4-23. Asymmetric sweep is still present. The weights also seem to be in agreement with some of the mapped faults. Injection from E14 seems to be confined to producer 349. This, in turn, deflects water from injectors E15, D14 and D15 away from this producer. In addition, fault A seems to be acting like a barrier to the south of injector C15, deflecting its...
injection northwards. This confines D15 injection to the SW-NE direction. The southern-most row of producers seems to be swept mainly from the south. Likewise, producers 369 and 319 are swept from the west.

**Figure 4-23** Chihuído de la Sierra Negra Field. Pseudo-allocation weights after omitting negative connections.
However, caution is advised regarding this interpretation. As previously mentioned, important information from wells outside the analysis window is not considered. For example F14 could turn out to be injecting only northwards. This would influence the weights associated with producer 218. Moreover, since we are applying a balance constraint, boundary conditions are very influential on the results.
This can be illustrated by eliminating the connection between C14 and a-192 based on the presence of the fault (and since the C15 results honor the faults). Figure 4-24 shows that C14 exclusively supports producer 352 as expected. However, now, the interpretation of the flow directions south of fault A changes. B15 which showed that 100% of its injection was flowing towards producer 355, is now supporting 100% of producer a-192.

Although this interpretation would seem more consistent with the interpreted fault, from an allocation model point of view, information from wells west of C14 and south of B15 would be required to come to a more solid conclusion.

Finally Figure 4-25 shows the dynamic behavior of the weights of producer 378 neighboring injectors. In spite of the (time average) balance condition, some
unexpected excursions into negative values can be observed for E16. Likewise, the graph shows some abrupt changes in allocation fraction whose possibility should be further investigated.

4.4.3. Rate Regression Results

Rate regression results for producers 379, 380, 349, and a-192 are presented in Figure 4-26, through Figure 4-29.

Producer 379 is one of four producers located near the center of the analysis window. It has full information from two successive rows of injectors. The rate data shows no abrupt change in operating conditions other than maybe between the 6th to 8th month of analysis.

Producer 380 also has full information from two successive ring of injectors. However, rate data suggests a probable change in BHP (pump) at month 22. Injection and production rates are balanced until month 15. Injection continues to increase, while the producer appears to be rate limited. The reservoir is presumably pressuring up. Based on the comparison of the actual and predicted production rates, the predicted rate trails the data by two months (the increase occurs in month 24 and not 22). This indicates that there is a two month lag between when the production rate is increased and the injection increases. By month 38, the rates are in balance again.
Figure 4-26 Chihuuido de la Sierra Negra. Model 4b rate regression results for producer 379.

Figure 4-27 Chihuuido de la Sierra Negra. Model 4b rate regression results for producer 380.
Figure 4-28 Chihuïdo de la Sierra Negra. Model 4b rate regression results for producer 349.

Figure 4-29 Chihuïdo de la Sierra Negra. Model 4b rate regression results for producer a-192
Wells 349 and a-192 represent lateral and corner producers respectively. Complete rate information for the second row of injectors is not available for these wells. Although the regression captures overall behavior, results are more scattered than those of producer 379 and 380.

4.5. CONCLUSIONS

The allocation model yields dynamic weights that account for interactions between adjacent and non-adjacent injectors. The time average of the weights has been shown to be equivalent to the tracer fraction allocation.

Very good results have been obtained in the smaller 5x4 Synfield. However the quality of the results was less satisfactory for the larger and more complex cases of the 25x16 Synfield. More research is needed to improve the model’s validation in complex environments. It is also not clear if tracer fractions are good indicators of injection allocation in this field.

Satisfactory results have been achieved for the Chihuído de la Sierra Negra dataset. This open boundary dataset has proved that the pseudo-allocation models are particularly sensible to spatially truncated datasets. On the other hand, the models also suggest that non-neighboring connections can be limited to the second row of injectors counting from the producer.

4.6. FUTURE WORK

1. Further research would be aided by streamline simulation models where injector-producer allocation is more readily visualized and accounted for.
2. The allocation models should be modified to account for BHP changes.

3. Alternatively, because injector interactions are captured in the B weight, these weights could be reformulated in terms of the difference in rates between the adjacent and non-adjacent injectors:

\[ B_{ji} = \beta_{i0} + \beta_{ii} I_i + \sum_{m} \beta_{im} (I_m - I_i) \]  

(4.29)

The summation term in Eq. (4.29) contains \( I_i \) terms, which suggests that the \( \beta_{ii} \) term might be dropped.

4. Also, a stricter balance could be proposed for the injectors. Instead of requiring the allocation weights to sum to one, the average allocated rate to the producers could be required to be equal to the injected rate. In other words, Eq. (4.22) could be replaced by:

\[ \sum_j E(B_{ji}I_j) = E(I_i) \quad \forall i \]  

(4.30)

The injection efficiency factor \( e_i \) is not required in Eq. (4.30) as it would cancel out.
CHAPTER 5: OIL PRODUCTION REGRESSION MODEL

The oil production regression model is based on a power law relationship between the instantaneous water-oil ratio (WOR) and the cumulative water injected (Wi). The approach taken is to superimpose this relationship onto the production models covered in Chapters 1 through 4.

We use this water-oil ratio power law relationship by replacing the cumulative water injected by the total fluids produced. For the latter we have already developed a number of models. Hence, the oil regression model builds on the total fluids produced model and, further, relies on its accuracy.

5.1.WATER OIL RATIO MODEL

The predicted water-oil ratio has the following form:

\[ WOR = a \cdot Wi^b \]  \hspace{1cm} (5.1)

After fill-up and once injection and production rates are in balance:

\[ Wi \approx Np + Wp = Qp \]  \hspace{1cm} (5.2)

where WOR is the instantaneous water-oil ratio produced.

Wi: cumulative water injected is approximated by the total fluids produced
Np is the cumulative oil produced [reservoir volumes].
Wp is the cumulative water produced [reservoir volumes].
Qp is the cumulative total fluids produced [reservoir volumes].
Note that neither i nor p are being used as a sub-indexes.

If the injection-production balance is achieved on a pattern-by-pattern basis, then expressions (5.1) and (5.2) can be applied for each producer. Hence, any of the multivariate linear regression methods can be used to obtain a total liquid rate prediction without explicit information regarding the origin of the injected water. Expression Eq. (5.1) can be linearized by taking base ten logs:

\[ \log(\text{WOR}) = A + b \log(W_i) \]  \hspace{1cm} (5.3)

where \( A = \log(a) \)

We now proceed to develop a linear regression based on this relationship to find \( A \) and \( b \) by minimizing the squared error. The resulting system in matrix form is:

\[
\begin{pmatrix}
1 & \text{E}[\log(W_i)] \\
\text{E}[\log(W_i)] & \text{E}[\log^2(W_i)]
\end{pmatrix}
\begin{pmatrix}
A \\
b
\end{pmatrix}
= \begin{pmatrix}
\text{E}[\log(\text{WOR})] \\
\text{E}[\log(\text{WOR}) \log(W_i)]
\end{pmatrix}
\]  \hspace{1cm} (5.4)

To apply system (5.4), \( W_i \) is first computed from numerical integration of the estimated producer rate \( \hat{P}_j \) estimated from MLR, ABMLR, IBMLR, etc.

\[ W_i \equiv \sum_t \hat{P}_j \Delta t \]  \hspace{1cm} (5.5)

where \( \Delta t \) is the sampling interval in units compatible with \( \hat{P}_j \).

Note that application of the system of equations (5.4) implies the following:

- Estimation of \( A \) and \( b \) is made for every producer.

- A single WOR-Wi power law relationship must apply throughout the entire regression time interval. Likewise, for prediction, it is assumed that this relationship will continue invariably in time.
-The variable regressed is log(WOR), which is distinct from the oil rate Qo or WOR. For example, a $\Delta \log(WOR) = 1$ from $\log(WOR) = -2$ to $\log(WOR) = -1$ represents a change in watercut fraction from 0.01 to 0.1. However, the same change in log(WOR) from log(WOR) = -1 to log(WOR) = 0, represents a change in watercut fraction ($f_w$) from 0.1 to 0.5. Recall that:

$$f_w = \frac{WOR}{WOR + 1} \tag{5.6}$$

Thus, the log transform implicitly weights early time data more than late time data.

Naturally, linearization of Eq. (5.1) comes at a price. Care must, thus, be taken when using this approach. It is suggested that only late time data be used for fitting and only if it shows a clear trend on a WOR-Wi crossplot. Otherwise, the use of non-linear optimization solvers is suggested to regress explicitly on the oil phase rate to avoid the logarithmic transform issues. More complex behavior of the WOR-Wi curve can be matched by resorting to allocation models in which the WOR is modeled with a superposition of WOR-Wi power laws; one for each injector adjacent to a producer. These systems are very flexible but not linearizable. This is illustrated in sections 5.4 and 5.4.2.

5.2. RELATIVE PERMEABILITY AND THE $b$ EXPONENT

The work by Yortsos et al. (1999) shows that if the ratio of the relative permeability of oil to water can be approximated by a power law relationship such as:

$$k_{ro}/k_{rw} \approx (1 - S_w - S_{or})^m \tag{5.7}$$

For late time behavior the WOR would follow:
\[
\log(\text{WOR}) \approx \frac{m}{1-m} \log(t_D) + H
\]
(5.8)

where \( H \) is a constant and \( t_D \) is a dimensionless injection time expressed as pore volumes injected.

\[
t_D = \frac{W_i}{V_p}
\]
(5.9)

where \( V_p \) is the pore volume.

Substitution of \( t_D \) by \( W_i \) in (5.8) amounts to a change in the value of \( H \).

Then, under appropriate conditions, the slope of a \( \log(\text{WOR}) \) vs. \( \log(W_i) \) plot can provide information on the exponent of the power-law dependence of the oil’s relative permeability on saturation.

By comparison of (5.8) to (5.3):

\[
b = -\frac{m}{m-1}
\]
(5.10)

Thus, matching of the oil phase rate through a WOR-Wi power law relationship would, potentially, result in information about the relative permeability ratio.

5.3. THE 5×4 HOMOGENEOUS SYNFIELD EXAMPLE

Figure 5-1 shows simulated water-oil ratios and cumulative fluids produced by producer P1 (see section 2.3.3 for simulation details). This is a well-behaved case where we can see that the power law relationship is constant for the whole range of WOR (Figure 5-1).
Figure 5-1. Producer P1 cumulative fluids produced vs. water-oil ratio (log-log scale). Power-law behavior with exponent \( b = 2.52 \) can be observed.

As seen in Figure 5-2, the regressed results for producer P1 are very good. In terms of the WOR, Figure 5-3 also shows almost perfect agreement \((R^2 > 0.99)\). Similar results are obtained for the other three producers.

Figure 5-2. Producer P1 oil phase regression results. Left: Oil rate vs. time. Data before breakthrough was not included in the regression. Right: cross-plot of oil rate vs. regressed oil rate.
Figure 5-3. Water-oil ratio cross-plot shows very good match between actual and regressed values.

5.3.1. 5x4 Synfield - The b exponent

As mentioned in section 5.2, under favorable conditions the value of the regressed b exponent might contain information on the relative permeability ratio exponent. This was tested for the 5x4 homogeneous Synfield.

Figure 5-4 (left) shows the relative permeability set input into the simulator for the 5x4 homogeneous Synfield case (section 2.3.1). The last three points of the input relative permeability table were transformed into the left and right terms of Eq. (5.7) and plotted Figure 5-4. A power-law relationship was fitted to this data to obtain slope m.
From Figure 5-4 right the value for $m$ (5.7) is 1.74. Applying the relationship given by Eq. (5.10):

$$b = \frac{m}{m - 1} \text{ or } b = \frac{1.74}{1.74 - 1} = 2.35$$

(5.11)

This value compares favorably with the slope of 2.52 in Figure 5-1. Hence, under favorable conditions, information is retrieved regarding the shape of relative permeability ratios.

5.4. PSEUDO ALLOCATION MODELS AND THE OIL MODEL

In the previous section a single water-oil ratio vs. cumulative fluids produced was coupled to a production rate estimate to obtain oil phase rate estimation. Another motivation for developing the high-order allocation models (see previous chapter) is shown in this section.
Assuming that the pseudo allocation models provide information on how the water is being allocated from the injectors to its neighboring producers, then a WOR-Wi power law can be applied to the individual volumes contributed by each injector to the producer after having solved for the allocation model B_{ji}:  

The total oil rate produced by producer j is the sum of the oil contribution from adjacent injectors:  
\[ \hat{P}^o_j = \sum_{i\text{Adjacent}} \hat{P}^o_{ji} \]

where the superscript o denotes oil phase.

The oil contribution from injector i to producer j is the total rate contribution times the injector-producer pair oil cut at that time level:  
\[ \hat{P}^o_{ji} = (1 - f_{W ji})\hat{P}_{ji} \]  
(5.12)

From the high-order allocation model the injector-producer pair rate contribution is:  
\[ \hat{P}_{ji} = B_{ji} I_i \]  
(5.13)

Water cut is calculated from the estimated water-oil rate as:  
\[ f_w = \frac{\text{WOR}}{1 + \text{WOR}} \]  
(5.14)

A power-law is used to model the injector-producer pair water-oil ratio as a function of the total fluids cumulative pair-wise contribution:  
\[ \tilde{\text{WOR}}_{ji} = a_{ji} \tilde{W}_{b_{ji}} \]  
(5.15)

The cumulative water injected into producer j’s drainage area by each injector is:  
\[ \tilde{W}_{ji} = \sum_{t=1}^{n} \hat{P}_{ji} \Delta t \]  
(5.16)
The objective function to be minimized is the sum of the squared residuals of
the oil rate Eq. (5.17). The sum is over the time levels, the regression variable and its
estimator are always at the same time level.

\[
\sum_{t} \left( P_{j}^{o} - \hat{P}_{j}^{o} \right)^{2}
\]  

(5.17)

Equation (5.17) is no longer linear in the regression parameters \(a_{ji}\) and \(b_{ji}\).

5.4.1. **Homogeneous 5x4 Synfield**

The oil rate contributions from each injector are shown for producer P1 in
Figure 5-5. The results show similar declining oil rates for the volumes being
contributed by injectors I1 and I2 and a lower oil rate contribution from injector I3.
This concurs with the fact that injector I3 feeds four producers instead of two and
thus its average allocated rate tends to be smaller than those of I1 and I2.

More interestingly Figure 5-6 shows a lower WOR corresponding to the
volume contributed by I3. The I3 contribution to WOR is not increasing as fast as the
contributions from I1 and I2. This is confirmed by the smaller water saturation
observed in the neighborhood of P1 from the volume being swept by I3 as opposed to
the larger water saturation observed from the volumes arriving from the I1 and I2
directions (Figure 5-7). In other words, both the physical and regression model show
a delayed behavior of WOR vs. time for I3 with respect to I1 and I2.
Figure 5-5. Producer P1 oil phase rate contributions from each adjacent injector. Left: linear scale. Right: semi-log scale.

Figure 5-6. Producer P1 water-oil ratio vs. time. Contribution from each of the adjacent injectors.
Figure 5-7. Simulated oil saturation distribution at the 30th timestep. The region around producer P1 shows larger water saturation from the volumes being swept by I1 and I2 than the volume being swept by I3.

Figure 5-8. Producer P1 actual oil rate vs. the estimated value.
Finally, the composite oil regression shows very good agreement with actual simulation results except for the initial part of the history (high oil rates) where WORs are very small (Figure 5-8).

5.4.2. The 5×4 Faulted Homogeneous Synfield

Recall section 2.3.3.c (Figure 2-18) and section 4.3.1 (Figure 4-16). The water-oil ratio for producer P1, in this case, exhibits a more complex behavior (Figure 5-9) than in the previous case. This is a result of the dissimilar flow geometries and breakthrough times between I2 and I5. In terms of relative permeability, this reservoir is homogeneous.

![Graph showing WOR behavior](image)

**Figure 5-9.** Faulted 5x4. Graph shows a more complex WOR behavior having an early time trend transitioning into a late time trend.

Based on this behavior, one might attempt a piece wise regression or, alternatively, only regress the early behavior or the late time behavior. Since the late
time behavior is usually more useful for prediction, we will show an example of the latter first.

a. Late time regression

The goal is to match the late time behavior observed in Figure 5-9 with a single water-oil ratio WOR vs. cumulative injected water Wi relationship. As mentioned previously, any of the reviewed estimators for Pj can be used to approximate Wi. In this case, the allocation model with non-centered moments and balance constraint has been chosen (section 4.1.6).

The second half of the data starting at day 1675 was used for regression purposes. Results are presented in Figure 5-10.

The WOR and oil rate are matched satisfactorily partly because the late-time WOR behavior can be matched with a single power law relationship.

![Figure 5-10 Oil rate regression results for late time.](image)
b. *Early time regression*

Using the same procedure but regressing on the first half of the data, day 76 to day 943, results in a satisfactory match for the early time WOR (Figure 5-11).

![Figure 5-11. Oil phase regression results for early time.](image)

As we have seen, the behavior of the WOR curve cannot be matched with a single two-parameter power law relationship in this case (nor with a single exponential decline curve). However, a combination of these might prove appropriate. As mentioned, pseudo allocation models are useful for this purpose.

c. *Overall Regression with Pseudo Allocation Models*

To illustrate the flexibility provided by the pseudo allocation models, an example follows that uses the high-order allocation model. In this case we will be making use of the cumulative injection volume allocated to the producer by each adjacent injector.

Results are presented for producer P1 in Figure 5-12. A good match for early and late time is achieved with a root mean squared error (RMSE) of 24 BOD.
\[
\text{RMSE} = \sqrt{\frac{1}{N} \sum_{i}^{N} \left( P_{ij}^{o} - \hat{P}_{ij}^{o} \right)^{2}}
\] (5.18)

The corresponding WOR parameters \(a_{ji}, b_{ji}\) and average pseudo-allocation factor \(B_{ji}\) for producer P1 are presented in Table 5-1. Individual pair, composite, and historic WOR-cumulative injected water are plotted in Figure 5-13.

**Table 5-1.** \(a\) and \(b\) WOR power-law parameter regression results for producer P1 pairs.

<table>
<thead>
<tr>
<th></th>
<th>I1</th>
<th>I2</th>
<th>I3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_{1i}) [bbl/bbl]</td>
<td>3.61E-08</td>
<td>7.28E-11</td>
<td>1.74E-10</td>
</tr>
<tr>
<td>(b_{1i}) [dimless]</td>
<td>2.18</td>
<td>2.35</td>
<td>2.70</td>
</tr>
<tr>
<td>(E[B_{1i}])</td>
<td>0.08</td>
<td>0.56</td>
<td>0.08</td>
</tr>
</tbody>
</table>

**Figure 5-12.** Oil phase regression results using pseudo allocation model 4b. Early and late time trends are matched. MSE is 24 BOD.
Figure 5-13. Regressed individual injector-producer pair, composite and historic data WOR vs. Wi.

Note that, since the model was constructed assuming the absence of faults, the allocation to P1 from I1, I2, I3 is by construction:

\[ P_1 = B_{11} I_1 + B_{12} I_2 + B_{13} I_3 \]  \hspace{1cm} (4.3)

Thus, P1 lacks allocation from I5. This is physically incorrect and is the result of ignoring the presence of the fault in the model. The example is still appropriate to show the results of combining the response from three injectors (I1, I2, I3) each with separate WOR power-law relationships.

5.4.3. Overall Regression with Pseudo Allocation Models acknowledging presence of the fault.

If the presence of the fault is acknowledged, then the allocation model becomes simpler for producer P1:

\[ \hat{P}_1 = B_{12} I_2 + B_{15} I_5 \]  \hspace{1cm} (5.19)
Moreover, there are no non-adjacent wells for P1 and thus the high-order allocation model becomes equivalent to the simplified allocation model (section 4.2). Good regression results for producer P1 total rate are obtained (Figure 5-14). Pseudo allocation weights B_{12} and B_{15} and their time behavior are shown in Figure 5-15.

![Figure 5-15. P1 total rate regression results for model 4b (presence of fault acknowledged).](image)

Oil contribution from injectors I2 and I5 are obtained by fitting the corresponding power-laws:

\[
\text{WOR}_{12} = a_{12} W_{12}^{b_{12}}
\]

\[
\text{WOR}_{15} = a_{15} W_{15}^{b_{15}}
\]

Results for a_{12}, a_{15} and b_{12}, b_{15} are presented in Table 5-2. The b_{ji} slopes do not differ significantly from those in Table 5-1 and these in turn agree with the value of 2.35 found in section (5.3.1). Figure 5-16 shows a good match, both for the WOR and for the oil rate. A slightly better match was achieved by acknowledging the fault. In this case the mean squared error is 17 BOD vs. 24 BOD in the previous case (Figure 5-12).
Finally, Figure 5-17 and Table 5-2 reveal that WOR behavior can be composed by the combination of two power-laws that share the same slope. This might also be an indication of relative permeability homogeneity.

![Graph showing allocation fraction vs. time for two different injectors, B12 and B15, with annotations indicating time in months and allocation fraction values ranging from 0.2 to 0.55.](image)

**Figure 5-15.** Producer P1 pseudo-allocation behavior vs. time (presence of fault acknowledged)

**Table 5-2.** a and b WOR power-law parameter regression results for injector-producer P1 pairs. Fault presence is acknowledged.

<table>
<thead>
<tr>
<th></th>
<th>I2</th>
<th>I5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{i}$</td>
<td>2.61E-09</td>
<td>2.84E-08</td>
</tr>
<tr>
<td>$b_{i}$</td>
<td>2.01</td>
<td>2.02</td>
</tr>
<tr>
<td>$E[B_{i}]$</td>
<td>0.49</td>
<td>0.32</td>
</tr>
</tbody>
</table>
Figure 5-16. Oil phase regression results acknowledging the presence of the fault. Early and late time trends are matched. MSE is 17 BOD.

Figure 5-17 Regressed individual injector-producer pair, composite and historic data WOR vs. Wi. Similar slopes suggest homogeneity.

These last two cases remind us that the non-linear regression solutions are non-unique. It is particularly important to have the regression model as consistent as possible with the known geologic features.
5.5. APPLICATION TO CHIHUIDO DE LA SIERRA NEGRA FIELD.

In this section the WOR power-law model is combined with the results obtained from the high-order allocation model for the ChSN field to obtain oil phase regressions. Oil phase rate results are shown in Figure 5-18, Figure 5-20, and Figure 5-22 for producers 379, 349, and 380. These graphs are followed by the water cut corresponding to the neighboring injectors contribution versus time in Figure 5-19, Figure 5-21, and Figure 5-23.

Figure 5-19 indicates that the match achieved in Figure 5-18 is caused exclusively by E16’s oil contribution; the other injectors either are not contributing to the total rate (E17, D16), as determined in section 4.4.2, or have completely watered out (D17). The hump in oil rate from month 6 to month 8 is attributed to a change in the operating BHP as discussed in the previous chapter. The change in the slope around month 24 corresponds to a change towards a positive slope in the total rate (see Figure 4-26).

Producer 349 shows a sharp decline in oil rate that transitions into a gentler decline after month 8 (Figure 5-20). Regression attributes the sharp decline to an abrupt watering out of E15’s contribution after which E14 remains as the sole contributor of oil for the remaining time (Figure 5-21).

Likewise, producer 380’s sole contributor of oil seems to be injector C16 (Figure 5-22 and Figure 5-23).

The oil contributions seem to be associated with small weights, which can be interpreted as a slower sweep from these injectors-producers pairs. This also implies that the swept areas between injectors-producers pairs with large allocation weights,
and thus larger pore volumes injected, might be watered out. On the other hand, E14 allocates all its water to producer 349 and still seems to be its main oil contributor (Figure 5-21). This contradiction is explained by the presence of faults B and C, which block flow to producer 218, effectively making the available swept volume between E14 and well 349 to be above average. However, as discussed previously, caution must be exercised when interpreting boundary wells.

**Figure 5-18** Producer 379. Coupled WOR power-law model and model 4b. Oil rate data and regression.
Figure 5-19 Producer 379. Water cut profiles from adjacent injectors contribution rates $q_{ij}$.

Figure 5-20 Producer 349. Coupled WOR power-law model and model 4b. Oil rate data and regression.
Figure 5-21 Producer 349. Water cut profiles from neighboring injectors.

Figure 5-22 Producer 380. Coupled WOR power-law model and model 4b. Oil rate data and regression.
Figure 5-23 Producer 380. Water cut profiles from adjacent injectors contribution rates.

5.6. CONCLUSIONS

Water-oil ratio versus cumulative volume power-law relationships have proven useful for fitting oil phase rate data in synthetic reservoirs as well as in field applications. In combination with the high-order allocation models they show flexibility in dealing with more complex oil rate declines.

The use of oil phase data proves a useful validation of the total rate model being used. If a satisfactory oil phase match is not achieved, the total rate model is flawed. The opposite, unfortunately, is not necessarily true. A combined match of the total and oil phase rate adds value to the interpretation of the field behavior.
In the examples shown in this chapter, oil phase regression was performed after regressing the total rate model. It is also possible to solve for both simultaneously by minimizing a combined weighted objective function.

More research is suggested with heterogeneous relative permeability reservoirs to better assess the relative permeability ratio information content of the regression exponent b.
Although reservoir engineers acknowledge the non-uniqueness of their solutions, sometimes they overlook the fact that the goodness of their prediction is not guaranteed by the goodness of the history match. Two models that explain past performance equally well might not explain future performance with the same degree of accuracy.

Reservoir characterization maximizes the use of the, often indirect, acquired data and makes inferences about reservoir characteristics at unsampled locations (mapping) at an arbitrary scale (e.g. a reservoir simulation cell). If this characterization explains past performance, it is usually accepted. In most cases, however, the model must be “tweaked” to achieve a good match to past performance. Sometimes these perturbations are arbitrary and/or locally disregard the geologic consistency of the model.

A recent paradigm change in reservoir characterization is the shift from deterministic to stochastic reservoir representations. These stochastic processes can be viewed as a series of images of the reservoir that honor (to a more or less degree) data at observed locations and that are considered equally probable. We refer to these images as constrained realizations.
Since this is only a conceptualization that accounts for our uncertainty and since the “true” reservoir is determined to be one and only one, these realizations are no longer equally probable the moment the reservoir starts producing and more data is gathered in the form of historic production data. Now the realizations must be sifted to find the ones more likely to be right according to the new information. This is one of current struggles of reservoir engineers and geoscientists involved in reservoir characterization: how to constrain stochastic models to production data.

6.1. GENERAL PROCEDURE

A direct way of ranking the realizations of a given geostatistical model is by submitting each realization to a reservoir simulator and then ranking the dynamic responses on the basis of a given objective function. A measure of error or goodness can be computed by comparing the results of the simulation to historic data. This approach is computationally very intensive. Streamline simulators have been championed as best suited to this task because of their faster execution times under favorable conditions compared to finite difference simulators (Ates (2003), Gross (2004), Mishra (2002), Wen (1998), Batycky and Thiele (2000)). Figure 6-1 summarizes this workflow where an optional final manual fine-tuning of the model step is also shown.
Figure 6.1 A simplified schematic showing one traditional approach to geostatistical model ranking.

In this section the use of weight ratio expression is proposed as a measure of local precision or suitability of a stochastic realization. Calculation of $R_{k_{ji}}$ is much faster than flow simulation. This section details how $R_{k_{ji}}$ can be used as a surrogate for flow simulation. From chapter 2:

$$R_{\beta_{ji}} = \frac{\beta_{ji}}{\sum_{\text{Pattern}} \beta_{ji}}$$  \hspace{1cm} (2.11)

$$R_{k_{ji}} = \frac{k_{ji}}{\sum_{\text{Pattern}} k_{ji}}$$  \hspace{1cm} (2.13)

where $k_{ji}$ is the average permeability between injector i and producer j calculated explicitly from the permeability field realizations.

In principle, the driving idea is very simple. Given the $\beta_{ji}$ instantaneously balanced multivariate regression (IBMLR) weights from a given field, the weight
ratios can be computed making use of expression Eq. (2.11). For each realization of a (or any number of) geostatistical model, weight ratios can be computed according to expression Eq. (2.13) (section 2.1.1.b) after estimating average permeabilities between neighboring injector-producer pairs. We will refer to these as weight ratio proxies. A measure of the average error can be computed between the set of weight ratios obtained from the regression weights and the proxies. Those realizations with the smallest average errors can be considered to be more likely to represent the actual permeability distributions and would, thus, result in flow simulations that more closely match historic rate data than those with high errors. Hence, the average error between the actual regression weight ratio and its proxy can be used as a criterion to rank geostatistical models without the need of numerical simulation. The whole procedure can be considered a substitute for an otherwise computationally intensive flow simulation-based procedure. This alternate procedure is summarized in Figure 6-2.

This procedure is aimed at capturing the permeability field and the reservoir dynamic response to total fluids produced. A match on phase ratios and pressures (volumes) would need to be achieved by other means.
### 6.2. LOCAL REALIZATION ACCURACY

The procedure described so far ranks realizations according to their overall semblance to the real distribution condensed, presumably, in the $R_{\beta_{ji}}$ weight ratios. However, for a given geostatistical model, some realizations might represent a subset of the reservoir better than others. This is true even when the conditioning data is regularly spaced. In other words, it is easier to find a realization that accurately describes a small region of the reservoir than the entire reservoir.

Generally speaking, procedures, such as the one presented in Figure 6-1, which are based on the goodness of a match to production data, are intrinsically limited in their ability to identify favorable “patches” within a realization; they rely on finding that elusive realization that is accurate all around.
On the other hand, expression Eq. (2.13), which describes the contrast in permeability around an injector, opens the possibility of evaluating subsets of the realizations locally around each injector. Hence, the procedure summarized in Figure 6-3 is proposed.

Just as in the previously described procedure, the permeability ratio proxy $R_{k_{ji}}$ is used to rank the set of realizations. Then the best-ranked realizations are loaded into a reservoir simulator to obtain the corresponding production response. The resulting production-injection datasets are regressed to obtain IBMLR weights and from these the weight ratios $R_{\beta_{ji}}$ are calculated.

The next step is to choose those injectors and realizations whose weight ratios compare favorable to the weight ratios calculated initially from the real history dataset. Presumably, the chosen realizations are accurate in terms of describing the average permeability contrast around the chosen injectors. Hence, the corresponding permeability fields around the selected injectors can be used as conditioning data for a second generation of realizations that would preserve desired features and would be geostatistically consistent. In other words, the semivariogram model is honored. However, it is acknowledged that this conditioning data is itself uncertain.

The same treatment can be repeated on this second generation of realizations until an appropriate weight ratio error is achieved.
Generate $n$ equiprobable models

Permeability ratio proxy

Rank models

Run best models

Regress best models and obtain weight ratios

Select injectors with good weight match

Use permeability field around these injectors as conditioning (frozen) data

**Figure 6-3** Weight ratio driven selection of most appropriate geostatistical realization.

Note the following:

- To avoid heavy computing, the first part of the procedure relies on the permeability ratio proxy. Simulation is still used on a reduced set of realizations.

- The weight ratios dictate the contrast in average permeability around an injector while the semivariogram dictates the characteristics of the spatial distribution of permeability values.

- A significant advantage of using this procedure, or variations thereof, is that not only is it able to rank geostatistical realization with less processor use, but it goes beyond the initial set of realizations into subsequently refined generations of consistent geostatistical models. On the other hand the number of total realizations necessary increases.
-It is presumed that this procedure is able to capture the large scale trends whereas purely production fitting is significantly sensitive to changes in the vicinity of the producers. Under the proposed scheme, this could be considered a final fine-tuning of the model after the large-scale variations have been captured.

-By building successively on local analysis it would seem that the often questionable inference of the mid to longer ranges of the semivariogram would be less critical to the effect of obtaining an overall correct permeability model. (see example II, section 6.3.4)

An alternate route that does not involve reservoir simulation is also shown in Figure 6-3 with a dotted line. Instead of resorting to reservoir simulation this step can omitted and the permeability ratio is compared instead to the individual weight ratios obtained from the historic data.

The method discussed is illustrated with two examples in the following sections.

6.3.SYNTHETIC APPLICATION EXAMPLES

Well layout and gridding are those corresponding to the 25x16 Synfield. The two-dimensional permeability distribution is geostatistically modeled using GSLIB’s sequential indicator simulation code SISIM for two facies. Each facies has a constant permeability; 300 md (good) and 50 md (poor) coded in red (dark) and yellow (light) colors, correspondingly, throughout the examples. Also, each facies occupies about half the reservoir volume.
The governing isotropic semivariogram is shown in Figure 6-4. A spherical model, with small nugget was used. The range of the spherical semivariogram is 1000 ft; slightly less than the well spacing of the 25x16 Synfield.

Figure 6-4 Normalized semivariogram used for the synthetic examples.

6.3.1. Weight Ratios Behavior Example

Before showing a complete example of how the procedure works, we give an example of how weight ratios can be used to locally assess the realizations. Figure 6-5 (left) presents a base case for which a reservoir simulation has been run and its results used to calculate weights via IBMLR. Of the 2000 well data conditioned simulations, Figure 6-5 (right) shows one that has been identified as good in terms of average error between weight ratio proxy and actual weight ratios of the base case. Generally speaking, the chosen realization is a visually good approximation to the base case. However, individual analysis of the weight ratios reveals that those
injectors whose weight ratio proxies are in closer agreement to those corresponding to the base case show a better match to the base realization around said injector and vice versa. Examples of patterns that show good agreement are identified in cyan (light) boxes whereas those that show disagreement are identified in blue (dark) color in Figure 6-5. Details of the region around injector I14 and I19 are shown in Figure 6-6 and Figure 6-7, respectively.

![Figure 6-5](image)

**Figure 6-5** Left: Base Case. Right: Good approximation as identified by average error in weight ratio proxies. Good agreement are identified in cyan (light) boxes whereas those that show disagreement are identified in blue (dark) color boxes. Refer to Figure 2-20 for detailed well location.
Figure 6-6 Weight ratio proxy for injector I14 (right) shows good agreement with base case weight ratio (left).

Figure 6-7 Weight ratio proxy for injector I19 (right) shows poor agreement with base case weight ratio (left). High permeability facies development is truncated between I19 and P15.
Table 6-1  Base case weights vs. weight ratio proxy. Left: values corresponding to I14 on Figure 6-6. Right: values corresponding to I19 on Figure 6-7. Weight mismatch can be observed for I19-P15.

<table>
<thead>
<tr>
<th>Base</th>
<th># 582</th>
<th>Base</th>
<th># 582</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
<tr>
<td>I14</td>
<td></td>
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<td>-</td>
<td>P16</td>
<td>0.33</td>
</tr>
</tbody>
</table>

Finally, Figure 6-8 shows cross-plots of base case weight ratios to weight ratio proxies for two arbitrarily chosen realizations. Based on the previous discussion, the realization corresponding to the graph with less scatter (on the right) should resemble the base or truth case more closely.

Figure 6-8 Cross-plot of base case weight ratio vs. weight ratio proxy for two realizations.
6.3.2. Procedure Application Example I

Making use of the semivariogram model described in section 6.3, unconditional sequential indicator simulations were run and the facies realization shown on Figure 6-10 (top most left) was randomly chosen as a surrogate for a real case. It will be referred to as base or truth case throughout this section.

Well facies data from the base case is used to construct a set of conditional realizations. For this example, we are assuming we have perfect knowledge of the underlying semivariogram structure. Nonetheless, it is not easy to come across realizations similar to the base case. The reasons being: i) the short range of the semivariogram, ii) some well locations are close to facies boundaries in the base case. In conditional simulations the data locations are in the middle of a facies pattern and away from facies boundaries.

In fact the percentage of locations that matched the facies present in the base case averaged 61% for this set of realizations (Generation 0). Some of these realizations along with the base case are shown in Figure 6-10.

The weight proxy error is calculated according to equations. (6.1) and (6.2):

\[ \varepsilon_i = \frac{1}{J} \sum_{j} \left( R_{k_ji} - R_{\beta_{ji}} \right)^2 \]  \hspace{1cm} (6.1)

\[ \varepsilon_R = \frac{1}{I} \sum_{i} \varepsilon_i \]  \hspace{1cm} (6.2)

where:

\( \varepsilon_i \) is the average weight ratio proxy for injector I

\( \varepsilon_R \) is the average weight ratio proxy error for a given realization.

\( R_{k_{ji}} \) is the weight ratio proxy calculated from the permeability realization.
$R_{\beta_{ji}}$ is the weight ratio from actual or historic data regression weights.

In Figure 6-11 the top performers in terms of the minimum weight ratio proxy error ($\varepsilon_R$) and minimum production rate error are shown for comparison. For the latter, 2000 flow simulation runs had to be performed and analysed.

For the flow simulations the production rate error is calculated as:

$$\varepsilon_S = \sum_j \sum_t (P_j - P_j^s)^2$$

(6.3)

where

$\varepsilon_S$ is the total production error for a given simulation.

$P_j^s$ is the simulated total liquid rate for producer $j$.

The average permeability input into the weight ratio proxy Eq. (2.13) as used throughout the examples presented in this chapter is calculated as follows.

The injection patterns such as the one depicted in Figure 6-9, are symmetrically divided into regions or volumes according to the injector-producer pair locations. It is assumed that most of the flow between an injector and a producer will occur through these regions. Cell permeabilities are arithmetically averaged within these regions and the resulting value is harmonically averaged with the permeability at the corresponding producer location Eq. (6.4). The resulting permeabilities are used in Eq. 2.13 to obtain the weight ratio proxy.

$$\bar{k}_{ji} = \frac{2}{\frac{1}{k_{V_{ji}}} + \frac{1}{k_j}}$$

(6.4)

where

$\bar{k}_{ji}$ is the average permeability input to the weight proxy expression Eq. (2.13).

$k_j$ is the permeability at producer $j$ location.
$k_{vji}$ is the arithmetic average of the permeability of the cells located within the volume determined by producer j and injector i. These regions are represented by the numbers 1 through 4 in the example Figure 6-9. The well cells are not included in these regions.

![Figure 6-9](image)

**Figure 6-9** Example of how the injection pattern is subdivided in order to compute average the permeabilities used in the weight ratio proxy expression.

Although expression Eq. (6.4) has given good results, as illustrated by the examples presented in this chapter, permeability averaging or upscaling has been a subject matter of active research (Renard, 1997), and thus, other averaging techniques might prove more appropriate than expression Eq. (6.4). In particular, the alternate average technique may include directional effects (such as arithmetic-harmonic averaging). In any case, it would seem that permeability at the producer location should be heavily weighted.
Figure 6-10 Although realizations are conditioned to well data, the short range of the semivariogram results in realizations that show different facies continuity or well connectivity. Top left: realization chosen as base or true case with IBMLR weights.
Following the proposed procedure, we will try to improve the realizations by identifying injectors that give the best weight ratios and preserve the local realization around them in the next generation of realizations (Generation 1). This can be implemented in several of ways. If the distance between selected injectors is larger than the range of the variogram, then local information from different realizations can be combined as conditioning data without risk of violating the semivariogram model. Otherwise a single contiguous subset of realizations, involving one or more injectors, would be required.

Generally speaking, best results are obtained when combining the weight proxy error of more than one adjacent well at a time.

For the present example, injectors I7, I9, I17 and I19 were chosen. The corresponding conditioning data and four realizations chosen randomly are presented in Figure 6-12.
Some features of the base case are starting to be reproduced more frequently and more closely after one generation. This is clear in Figure 6-13 where the best realization, according to the minimum weight ratio proxy error, is shown on the left.

For comparison, all 2000 realizations were run on Eclipse; the one that resulted in the minimum production error is shown on the right. Both realizations show significant improvements over the best available original realizations shown in Figure 6-11.
The process is continued until no major improvement in the weight ratio error is achieved. In this case, it was repeated until a 9th generation where 23 out of the 25 injector neighborhoods were used as conditioning data or “frozen”. The final result is shown in Figure 6-14 for both, the realization with the best weight ratio proxy and the best according to production error. The resemblance to the best case is remarkable (Figure 6-10 topmost left).

Figure 6-13 Generation 1. Left: best realization according weight ratio proxy RMSE 122 bbl/d. Right: best realization according to minimum production rate error RMSE 66 bbl/d.

Figure 6-14 Generation 9. Left: top realization according weight ratio proxy. Right: best realization according to minimum production rate error.
Semivariogram reproduction is also satisfactory (Figure 6-15). Note that even some ergodic fluctuations of the base case were also picked up by the final realization. Within the semivariogram range (1000 ft), the base case and best match agree very closely. For intermediate ranges (1000-2000 ft) there is some discrepancy, and finally structures at lag distances greater than 2000 ft are reproduced adequately.

Figure 6-15 Semivariograms of base case and of best realization (weight ratio). Left semivariogram in X direction. Right: semivariogram in Y direction.

As the process advances from generation to generation, it becomes more and more deterministic. However, there some ways to counterbalance this:
- The process can be restarted from a different “seed” number for generation 0.
- Local realizations to be preserved can be selected from sampling the top 5% (for example) within the rank.
- As generations advance, previously frozen realizations with less than optimal weight ratio can be dropped or unfrozen.
- A variance can be assumed or calculated for the weights and the weights resampled (and renormalized to sum to one) accordingly after each loop. This has the advantage of acknowledging the degree of uncertainty in the regression.

6.3.3. Procedure Evolution

Since this example is synthetic, it is possible to keep track of the evolution of the model as it progresses through the generations. In particular we can monitor how the objective function, the average weight ratio proxy error $\varepsilon_R$ Eq. (6.2), behaves with respect to how well the facies distribution is reproduced and also with respect to a measure of production rate fit.

Figure 6-16 shows the weight ratio proxy total error versus the fraction of cells whose facies match that of the base case. Results for generation 0, which represents classical stochastic simulation conditioned only to well data, generation 5, and generation 9 are presented. Each generation contains 2000 realizations.

For all generations, a trend to higher degree of semblance to the base case is observed as a smaller error (or better match to the weight ratio) is achieved. However, their domain is clearly capped. The base case seems to be an extreme outlier for generation 0 even though the semivariogram is correct and the realizations are constrained to well data. That there is dispersion along the y-axis, that is to say for a given weight ratio error there are different degrees of semblance to the base case, can be explained by the non-uniqueness of the averaging or upscaling technique Eq. (6.4). As mentioned in the previous section, it is expected that if a more appropriate
upscaling technique is implemented (particularly one which differentiates directional distributions), the dispersion would be lower.

![Graph showing weight ratio proxy error versus fraction of cells that match the base case for different generations.](image)

**Figure 6-16** Weight ratio proxy error versus fraction of cells that match the base case for different generations.

As locally correct realizations are preserved from generation to generation, both the average weight ratio error (x-axis) decreases and the semblance to the base case increases (y-axis).

However, since invariably there will be some error in the local realizations chosen to be preserved, there will be an upper limit to the semblance match and a lower limit in the weight ratio match (also recall that a proxy is being used).
Figure 6-17 Weight ratio proxy error versus simulated production error for different generations.

A similar behavior is observed in Figure 6-17 where the weight ratio proxy error (x-axis) is plotted against the root mean squared error of the production data (y-axis). Generally speaking, the smaller the error in the weights, the smaller the error in the production rates. However, the graph also suggests that as generations of realizations progress, initially there is a decrease in the mean of the production error, but after some point (about 5 generations) it stabilizes. At the same time the variance of the production error (vertical dispersion in the graph) does not decrease significantly even for the last generation, which has the most semblance to the base case. One would expect that as the realizations become more accurate, the production error and its dispersion should decrease. Nonetheless, this variability is the result of perturbations in the vicinity of the producers. This in turn suggests that the final
models can be history matched or fined tuned by relatively minor modifications of permeability in the immediate vicinity of the producers.

6.3.4. Example II

The previous case it was assumed that the underlying semivariogram model was perfectly known. In this example, a practical limitation in the inference of the semivariogram model is acknowledged. The goal is thus to illustrate the potential benefits of using the weight ratio analysis to guide geostatistical realizations when there is uncertainty regarding the inference of the semivariogram model.

Figure 6-18 Arbitrary zigzagging channel-type facies distribution. IBMLR weights have been overlain. Channel facies, in red (dark) permeability is 120 md. Background permeability is 40 md.
Figure 6-18 shows an arbitrary path of a high permeability channel cutting through a lower permeability background. The channel runs through well locations. Inference of a semivariogram in this case is not trivial. Furthermore, it would depend on the orientation of the principal anisotropy axis chosen as can be observed in Figure 6-19. The experimental semivariograms in the x and y (0 and 90 degrees), plotted on the left, differ from the experimental semivariograms in the 45 and 135 degree directions plotted on the right.

It is proposed that a simple short-range semivariogram in combination with the weight proxy ratio technique described in the previous section 6.3.2 be used instead. Moreover, the same isotropic spherical semivariogram with 1000ft range used in the previous example will be used. This kind of semivariogram will produce “blobs”. The location of these “blobs” will be dictated by the location of well conditioning data. It is hoped that the weight ratio will inform regarding which pattern of “blobs” will generate a connectivity similar to that of the base case (Figure 6-18).

Figure 6-20 shows four randomly chosen conditional realizations from the 2000 generated (generation 0). While some realizations capture the base case facies distribution, most fail in generating the connectivity of the base case.

As in the previous case the 2000 conditional realizations were ranked according to weight ratio proxies and to average production error calculated from reservoir simulation output. The top rankers according to each criterion are shown in Figure 6-21. In both cases the right connectivity was retrieved. In this case there was no need to advance to further generations.
For the best weight ratio proxy realization, individual weight ratio errors can be analyzed. Figure 6-22 shows how, in addition to having identified a realization that captures the base case trend, a bad spot can be identified by the large error in I22. This can be traced to an erroneous presence of high permeability facies between I22 and P14 for the corresponding realization (Figure 6-21, oval on the right).

Finally, semivariogram reproduction is shown in Figure 6-23. The base or truth case experimental semivariogram is overlain with the experimental semivariogram corresponding to the case in Figure 6-21, right. Good reproduction is shown in both the 0, 90 degree and 45, 135 degree semivariogram sets.
Figure 6-20 Four randomly chosen conditional realizations. While some realizations are similar to the base case (bottom left and right), others exhibit undesired facies discontinuities (top left)
**Figure 6-21** Left: best realization according to minimum monthly production error. Right: best realization according to average weight ratio proxy error.

**Figure 6-22** Average weight ratio error for each injector for the case shown in Figure 6-21 right. Note that the large error in I22 can be traced to an erroneously large permeability between I22 and P14 (marked with an oval in Figure 6-22).
Figure 6-23 Example II experimental semivariogram. Base case versus best case. Top figures: 0 and 90 degree directions. Bottom figures: 45 and 135 degree directions. Good semivariogram reproduction is achieved.
6.4. FUTURE WORK

The workflow described in Figure 6-3 and illustrated in Examples I and II is a supervised procedure; user input and action is needed at almost every step. The encouraging results of Examples I and II suggest that it would be very handy to incorporate the injection-production information carried by the MLR weights into the geostatistical process such that the resulting realizations inherit observed dynamic behavior. In other words, the weight ratios could be used to constraint the realization to production data.

One way of doing this is through the use of simulated annealing (SA) as presented in Figure 6-24. Using the average permeability ratio as a proxy for the weight ratio, a mismatch to the base case weight ratio can be incorporated into the objective function of an annealing procedure.

This is not developed in this work but is suggested as a natural continuation of research on this subject.
1. Incorporate weight ratio mismatch in annealing objective function
2. Generate Models with SA
3. Rank Models
4. Run Best Models
5. Regress Best Models

Figure 6-24 Incorporation of dynamic data captured by the IBMLR weights into geostatistical realizations through simulated annealing.
CHAPTER 7: CONCLUSIONS AND FUTURE WORK

This research work was performed following Albertoni’s (2002) work. His work was based on three main hypotheses:

- It is possible to quantitatively determine the communication between wells in a waterflood using only production and injection rate data.

- Production rate in every producer can be predicted given the injection rate.

- The information about inter-well connectivity can be used to map reservoir heterogeneities, preferential permeability trends, and transmissibility barriers.

Following these hypotheses, Albertoni developed different statistical approaches based on constrained multivariate linear regression.

The present work focused on analyzing the quantitative physical information contained in the regression weights and addresses some of the limitation of the model such as the assumption of constant producing bottom-hole pressure. An expression that explicitly relates the regression coefficients, or inter-well connectivity, to the physical parameters that govern flow equations has been proposed and tested successfully in synthetic numerical reservoirs.

Geostatistical models can be used with regression coefficients to create permeability field images that better represent the characteristics of the reservoir. A permeability weight ratio proxy has been proposed to rank realizations without having to submit each realization to a reservoir simulator.
Moreover, variogram inference becomes less critical within the workflow and in some cases, simpler variogram models could be used. Short range description of the variogram, however, remains extremely important in reproducing the characteristics of the geology.

An oil phase regression model coupled to the production rates estimated by regression models has also been proposed. This has been tested in several numerically simulated fields and then applied to one waterflood in Argentina.

An injector to producer water allocation model that accounts for adjacent and non-adjacent injector interactions through high-order statistics has been proposed. This model describes how injectors distribute water into the waterflood pattern.

One major assumption throughout this work is that the reservoirs to be analyzed should have small compressibility. This is a fair assumption for mature waterfloods developed in small acreage pattern; however it precludes the use of results in a quantitative manner in fields with significant gas saturation and/or very large well spacing.

Most of the applications of the regression weights presented in this work are derived from a physical interpretation and rely on their quantitative nature. Hence it becomes critically important that the regressed weights be robust and as consistent with secondary knowledge as possible before attempting their use in a quantitative manner.

The detailed conclusions drawn from this thesis are presented in the following sections. The last section of this chapter presents recommendations for future work including further research on topics that remain unresolved and research on topics
that were not studied in this work but may give satisfactory solutions to the problems encountered.

7.1. CONCLUSIONS

7.1.1. Linear Regression Weights

1. The IBMLR system of equations represents a medium in which injectors independently contribute a fraction of their rate to the producers. The weights can be interpreted as the fraction of injected water that would flow from an injector to a producer if all other injectors were shut-in. This is also equivalent to the relative average transmissibility between an injector-producer pair divided by the sum of transmissibilities between said injector and all producers. This has been tested in one-dimensional, two-dimensional, homogeneous and heterogeneous reservoirs.

2. The instantaneously balanced multivariate regression (IBMLR) model is more robust than the multivariate linear regression (MLR) model with respect to datasets that have dissimilar magnitudes of variance in the injection rates.

3. The accuracy of the weights decreases proportionally to the measurement noise and to the inverse of the injector-injector covariance matrix.

4. The IBMLR is able to deal with injector shut-in periods as long as the shut-in periods of different injectors do not overlap.

5. By connecting neighboring well pairs whose IBMLR weight is greater than a certain cutoff with a line, hydraulically independent blocks can be identified in the reservoir.
7.1.2. Bottomhole Pressure Fluctuations

6. A generic model that incorporates bottomhole pressure information has been developed. This model overcomes the constant bottomhole pressure limitation of previous models. The injector-producer weights obtained are consistent with the IBMLR weights and their physical interpretation. In addition, the new producer-producer weights obtained also contain a quantitative physical meaning: they are the transmissibility between the two producers. Finally, the volume between producers is explicitly probed resulting in more areal coverage of the reservoir properties.

7.1.3. High-Order Allocation Models

7. Allocation models are a way of presenting high-order moments regression models results in a physically interpretable form by packing several static regression coefficients into a dynamic term whose behavior is akin to a tracer fraction allocation.

8. Very good allocation results have been obtained in the smaller 5x4 Synfield. However the quality of the results was less satisfactory for the larger and more complex cases of the 25x16 Synfield.

9. Interesting allocation results have been achieved for the Chihuido de la Sierra Negra dataset. This open boundary dataset has proved that the pseudo-allocation models are particularly sensible to spatially truncated datasets.

10. Non-neighboring connections in the allocation model can be limited to the two rows of injectors.
7.1.4. Oil Rate Regression

11. Water-oil ratio modeled as a power-law function of cumulative water injected has proven a simple yet effective way of obtaining oil phase rate regressions.

12. Combining the allocation model with the water-oil ratio model results in a powerful oil phase rate matching procedure which, under limited testing, has proven able to identify watercut contribution from each injector. In particular watered-out contributions could be identified.

7.1.5. Application to Geostatistics

13. The weight ratio vs. average permeability ratio comparison procedure has successfully been at identifying geostatistical realizations that conform to production-injection data.

7.2. FUTURE WORK

1. A sensitivity study on the boundary injector rates is suggested for the MLR, ABMLR, IBMLR as well as for the high order allocation model.

2. Alternatively, for open boundary media, an injection efficiency factor might be attached to the injector rate or included as a balance condition for boundary wells. This efficiency factor would be incorporated to the regression coefficient set. Thus the amount of water allocated inside the boundaries would be determined from regression. The system of equations would no longer be linear. E.g.:

\[ \sum_j \beta_{ji} = e_i \quad \text{for boundary injectors.} \]
3. Results from section 2.3.3.d suggest that the IBMLR might benefit from additional non-bias constraints for each producer. This might reduce the occurrence of significant negative weights across hydraulically independent blocks.

4. Space filtering before regression could be tested. One of such filters could be the inverse distance-weighting coefficient multiplying the injection rates. The filtered injection rates would be producer specific:

\[ I'_{ij} = \frac{1}{\sum_j \frac{1}{L_{ji}}} I_i \]

5. Another family of regression models that might successfully address the compressibility issue is the multivariate auto-regression models. These models include terms for a number (equal to the order of the model, \( p \)) of previous time-levels e.g.:

\[ \hat{P}_j = \beta_0 + \sum_{m=1}^{p} \beta^{t-m} I^{t-m} \]

where \( \beta \) is vector row of weights, \( I \) is column vector of injector rates, and \( t \) is a time level.

Producer bottomhole pressure terms can also be included.

6. Production regression models can be formulated based entirely on bottomhole pressures. E.g.:

\[ \hat{P}_j = \beta_0 + \sum_i \beta_{ji} \Delta P_{wj} + \sum_k \gamma_{jk} \Delta P_{wk} \]
where the first sum account for the pressure difference between each injector and producer \( j \) and the second sum accounts for the pressure difference between each producer and producer \( j \).

7. Allocation models should be tested against streamline simulation model results.

8. Simultaneous oil and total rate regression is possible by minimizing a combined weighted objective function.

9. Prediction capabilities of allocation and oil rate models should also be tested. In particular, a comparison with decline curve analysis would be of great value.

10. Incorporation of weight ratio proxy error into the objective function of a simulated annealing procedure could be a way of introducing dynamic reservoir behavior as determined by injection-production response into a geostatistical model or realization.
NOMENCLATURE

\[ A = \] Arbitrary real symmetric matrix
\[ A = \] Asymmetry coefficient, dimensionless
\[ a = \] Regression coefficient for water-oil ratio model
\[ A_{ij} = \] Area open to flow for the producer-injector pair, \( \text{ft}^2 \)
\[ a_{wxyz} = \] High-order allocation model injector-injector covariance matrix element
\[ b = \] Injected water regression coefficient exponent for water-oil ratio model
\[ b_i = \] Regression weight bias, dimensionless
\[ b_{ik} = \] High-order allocation model producer-injector covariance matrix element
\[ B_{ji} = \] High-order allocation model weight (factor), dimensionless
\[ BHP = \] Flowing bottomhole pressure, psia
\[ C_i = \] Sum of injector i weights, dimensionless
\[ C_{BHP-BHP} = \] Producer bottomhole pressure – producer bottomhole pressure covariance matrix
\[ C_{BHP-P} = \] \( N \times N_{\text{jvar}} \) long bottomhole pressure–production rate covariance vector
\[ C_{I-BHP} = \] Injector rate-producer bottomhole pressure covariance matrix
\[ C_{I-BHP}^T = \] Transpose of \( C_{I-BHP} \)
\[ C_{II} = \] Injector-injector covariance matrix, \( (\text{rb/d})^2 \)
\[ Cov() = \] Covariance \( (\text{rb/d})^2 \)
\[ C_{PI} = \] Producer-injector covariance vector, \( (\text{rb/d})^2 \)
\[ E() = \] Expected value
\[ e_i = \] Injection efficiency factor
\[ f_w = \] Watercut, fraction
\( I_i \) = Rate of injector \( i \), rb/d

\( \bar{I} \) = Vector containing the time average injection rates, rb/d

\( K \) = Cross-covariance matrix

\( k \) = Permeability, md

\( \bar{k} \) = Effective average permeability, md

\( k_j \) = Permeability at producer \( j \) location

\( k_{ro} \) = Oil-phase relative permeability, fraction

\( k_{rw} \) = Water-phase relative permeability, fraction

\( \bar{k}_{V,ji} \) = Arithmetic average of the permeability of the cells located within the volume determined by the well pair

\( L \) = Length of the flow path, ft

\( m \) = Saturation exponent, dimensionless

\( N \) = Total number of production wells

\( N_{jvar} \) = Number of producers with fluctuating bottomhole pressures

\( N_p \) = Cumulative oil produced, rb

\( P_j \) = Rate of producer \( j \), rb/d

\( \bar{P}_j \) = Time-average rate of producer \( j \), rb/d

\( \hat{P}_j \) = Estimated rate of producer \( j \), rb/d

\( \hat{P}_j^\circ \) = Estimated producer oil rate, bbl/d

\( \hat{P}_{ji}^\circ \) = Estimated injector-producer oil rate contribution, bbl/d

\( P_{wf} \) = Flowing bottomhole pressure, psia

\( q_{ji} \) = Rate contribution for injector-producer pair, rb/d

\( Q_p \) = Cumulative total fluids produced, rb

\( R^2 \) = Coefficient of determination

\( R_k \) = Weight ratio proxy, dimensionless
\( R_{\beta} = \) Weight ratio, dimensionless
\( S_{or} = \) Residual oil saturation, fraction
\( S_{w} = \) Water saturation, fraction
\( t_D = \) Pore volumes injected, fraction
\( \Delta t = \) Time sampling interval
\( T_{ij} = \) Transmissibility between injector i and producer j, rb/d / psia
\( \text{Var}(\cdot) = \) Variance (rb/d)^2
\( Wi = \) Cumulative water injected, rb
\( Wp = \) Cumulative water produced, rb
\( X = \) Random variable
\( Y = \) Random variable

\( \beta_{ji} = \) Allocation model injector interaction weight, dimensionless
\( \beta_o = \) Vector of error-free or true weights, dimensionless
\( \beta = \) Vector containing the weights, dimensionless
\( \epsilon_i = \) Average weight ratio proxy for injector I
\( \epsilon_R = \) Average weight ratio proxy error for a given realization
\( \epsilon_S = \) Total production error for a given simulation
\( \epsilon_o = \) Production rate measurement error vector, rb/d
\( \gamma_{jk} = \) IBMLR+P producer-producer weigh, rb/psia
\( \kappa(\cdot) = \) Conditioning number, dimensionless
\( \lambda_{\text{max}} = \) Maximum eigenvalue of matrix A, dimensionless
\( \lambda_{\text{min}} = \) Minimum eigenvalue of matrix A, dimensionless
\( \mu = \) Viscosity of the fluid, cp
\( \mu = \) Lagrange multiplier, (rb/d)^2
\( \overline{\mu} = \) Vector containing Lagrange multipliers, (rb/d)^2
\[ \rho = \text{Weighting coefficient, IFP model, dimensionless} \]
\[ \sigma_{ij}^2 = \text{Injector-producer covariance, (rb/d)^2 or (rm^3/d)^2} \]
\[ = \text{Production rate error variance, (rb/d)^2} \]

\[ \vec{1} = \text{Vector containing ones} \]
\[ \vec{I} = \text{Identity matrix} \]
\[ \vec{\theta} = \text{Matrix containing zeroes} \]

**Subscripts**

\[ i = \text{Injector index} \]
\[ h = \text{Injector index} \]
\[ k = \text{Producer index} \]
\[ j = \text{Producer index} \]
\[ i_{\text{Adjacent}} = \text{List of injectors adjacent to producer } j \text{ index} \]
\[ j_{\text{Pattern}} = \text{List of producers that belong to injectors’ } i \text{ injection pattern index} \]
\[ m = \text{List of injectors non-adjacent to producer } j \text{ index} \]

**Superscripts**

\[ m = \text{Observed data point} \]
\[ n = \text{Time} \]
\[ o = \text{Oil-phase} \]
\[ s = \text{Simulation} \]
Acronyms

\[ ABMLR = \] Average balance multivariate linear regression
\[ IBMLR = \] Instantaneously balanced multivariate linear regression
\[ IBMLR+P = \] Instantaneously balanced multivariate linear regression including flowing bottomhole pressure terms
\[ IDW = \] Inverse distance weighting
\[ IDWK = \] Inverse distance weighted permeability
\[ IFP = \] Injection from production
\[ MLR = \] Multivariate linear regression
\[ RMSE = \] Root mean squared error, bbl/d
\[ WOR = \] Instantaneous water-oil ratio produced, fraction
REFERENCES


VITA

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